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Civil Engineering
16/ENG03/027
ENG281

1) Evaluate the following limits of function

a)
$$\lim_{x \to \pi/2} \left[\frac{\left(x^2 - \frac{\pi}{4}\right) \sin(\cos x)}{x - \frac{\pi}{2}} \right]$$

<u>Solution</u>

$$\lim_{x \to \pi/2} \left[\frac{\left(x^2 - \pi/4\right) \sin(\cos x)}{x - \pi/2} \right] = \left[\frac{\left(\left(\frac{\pi}{2}\right)^2 - \frac{\pi}{4}\right) \sin\left(\cos \frac{\pi}{2}\right)}{\frac{\pi}{2} - \frac{\pi}{2}} \right]$$

$$= \left[\frac{\left(\left(\frac{\pi}{2}\right)^2 - \frac{\pi}{4}\right) \sin\left(\cos \frac{\pi}{2}\right)}{\frac{\pi}{2} - \frac{\pi}{2}} \right] = \frac{\left(\frac{\pi^2}{4} - \frac{\pi}{4}\right) \sin\left(\cos \frac{\pi}{2}\right)}{0} = \text{Indeteminant}$$

∴ Using l'hopital law : $\frac{dy}{dx}$ of the numerator $= u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{dy}{dx} =$ let $u = x^2 - \frac{\pi}{4}$ and $v = \sin(\cos x)$

$$\frac{dy}{dx} = 2x \quad \frac{dv}{dx} =$$

$\frac{d \sin(\cos x)}{dx} =$ let $\cos x = w$

$$v = \sin w$$

$$\frac{dv}{dw} = \cos w \quad ; \quad \frac{dw}{dx} = -\sin x$$

$$\frac{dv}{dx} = \frac{dv}{dw} \times \frac{dw}{dx} = -\sin x \cos(\cos x)$$

$$= \frac{\left(x^2 - \frac{\pi}{4}\right)x - \sin x \cos(\cos x) + \sin(\cos x)(2x)}{1}$$

$$= \cancel{\frac{(90^2 - 45)x - \sin 90 \cos(\cos 90) + \sin(\cos 90)(2\times 90)}{1}}$$

$$= \left(\frac{\pi}{2}\right)^2 - \frac{\pi}{4}x - \sin 90 \cos(\cos 90) + \sin\cos 90 \times 2\left(\frac{\pi}{2}\right)$$

$$= \left(\frac{\pi^2}{4} - \frac{\pi}{4}\right)x - 1 + 0\times\pi$$

$$= \frac{-\pi^2}{4} + \frac{\pi}{4} \quad ; \quad = \frac{\pi}{4} - \frac{\pi^2}{4} \quad ; \quad = \frac{\pi(1-\pi)}{4}$$

16) $\lim_{x \to \pi/2} \ln\left(\exp \frac{(3x^2+2x-1)}{x+1}\right)$

Solution

$$= \lim_{x \to \pi/2} \ln\left(\exp\left(\frac{3x^2+2x-1}{x+1}\right)\right) = \lim_{x \to \pi/2} \ln\left(\exp\left(\frac{(3x-1)(x+1)}{x+1}\right)\right)$$

$$= \lim_{x \to \pi/2} \ln(\exp(3x-1))$$

$$= \lim_{x \to \pi/2} (3x-1) = 3(\pi/2) - 1$$

$$= \frac{3\pi}{2} - 1 = \frac{3\pi}{2} - \frac{1}{1} = \frac{3\pi - 2}{2}$$

c) $\lim_{x \to 2+\sqrt{3}} \cos\left(\sin^{-1}\frac{(x-2)}{(x-\sqrt{3})}\right)$

Solution

$$\lim_{x \to 2+\sqrt{3}} \cos\left[\sin^{-1}\frac{(2+\sqrt{3}-2)}{(2+\sqrt{3}-\sqrt{3})}\right]$$

$$= \cos\left[\sin^{-1}\left[\frac{\sqrt{3}}{2}\right]\right] = \cos\left[\sin^{-1}(0.8660)\right]$$

$$= \cos 60 = \frac{1}{2}$$

d) $\lim_{x \to 4} \left[\frac{x^2-8x+16}{x^2-5x+4}\right]$

Solution

$$\lim_{x \to 4} \left[\frac{\cancel{(x-4)}(x-4)}{\cancel{(x-4)}(x-1)}\right]$$

$$= \lim_{x \to 4}\left[\frac{x-4}{x-1}\right] = \frac{4-4}{4-1} = \frac{0}{3} = 0$$

2) Determine whether each of the following series is convergent

a) $\frac{2}{2\times 3} + \frac{2}{3\times 4} + \frac{2}{4\times 5} + \frac{2}{5\times 6} + \cdots$

Solution

$$U_n = \frac{2}{(n+1)(n+2)} \; ; \; U_{n+1} = \frac{2}{(n+2)(n+3)}$$

Ration: $\frac{U_{n+1}}{U_n} = \frac{2}{(n+2)(n+3)} \times \frac{(n+1)(n+2)}{2}$

$$\frac{U_{n+1}}{U_n} = \frac{n+1}{n+3}$$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{n+1}{n+3}$$

$$\lim_{n\to\infty} = \left[\frac{n/n + 1/n}{n/n + 3/n}\right] = \frac{1+1/n}{1+3/n} = \frac{1+0}{1+0} = \frac{1}{1} = 1$$

Since $\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = 1$, the series is inconclusive which means it is either divergent or convergent

Using comparison test, if $U_n <$ P series

for P series, $U_n = \frac{2}{n^P}$

for the series, $U_n = \frac{2}{(n+1)(n+2)}$

Using 1, 10, 100

for 1, $\frac{2}{2\times3} < \frac{2}{1^2}$

for 10, $\frac{2}{11\times12} < \frac{2}{10^2}$

for 100, $\frac{2}{101\times102} < \frac{2}{100^2}$

Since U_n is less than P_{series} ∴ the series is convergent

2b) $\frac{2}{1^2} + \frac{2}{2^2} + \frac{2}{3^2} + \frac{2}{4^2} + \dots$

Solution

Using D'Alembert's ratio.

$U_n = \frac{2}{n^2}$

$U_{n+1} = \frac{2}{(n+1)^2} = \frac{2}{n^2+2n+1}$

$\frac{U_{n+1}}{U_n} = \frac{2}{n^2+2n+1} \times \frac{n^2}{2} = \frac{n^2}{n^2+2n+1}$

$\lim_{n \to \infty} \frac{U_{n+1}}{U_n} = \left[\frac{1}{1 + 2/n + 1/n^2}\right]$

at $n \to \infty$ $\left[\frac{1}{1+0+0}\right] = \frac{1}{1} = 1$

Since, $\lim_{n \to \infty} \frac{U_{n+1}}{U_n} = 1$, the series is inconclusive.

∴ It can be convergent or divergent.

~~Using comparison test,~~

~~$U_n = \frac{2}{n^2}$~~

~~for P series,~~

~~Using Test 1,~~

~~$\lim_{n \to \infty} U_n =$~~

Using Comparison test,

P series, $\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \dots$

Series given: $\frac{2}{1^2} + \frac{2}{2^2} + \frac{3}{3^2} + \frac{2}{4^2} + \cdots$

Using 1, 2, 3, 4

$P_{series} = \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$

Series given $= \frac{2}{1} + \frac{1}{2} + \frac{2}{9} + \frac{1}{8} + \cdots$

Comparision: for 1, $\frac{2}{1} > \frac{1}{1}$ and for 3, $\frac{2}{9} > \frac{1}{9}$

$\therefore$ The series is divergent.

2c) $U_n = \frac{1+2n^2}{1+n^2}$

Solution

$$U_n = \frac{1+2n^2}{1+n^2} \quad ; \quad U_{n+1} = \frac{1+2(n+1)^2}{1+(n+1)^2} = \frac{1+2(n^2+2n+1)}{1+n^2+2n+1}$$

$$= \frac{1+2n^2+4n+2}{1+n^2+2n+1} = \frac{3+2n^2+4n}{2+n^2+2n}$$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty}\left[\frac{3+2n^2+4n}{2+n^2+2n}\right] = \left[\frac{3/n^2+2+4/n}{2/n^2+1+2/n}\right]$$

$$= \frac{0+2+0}{0+1+0} = \frac{2}{1} = 2$$

Since $\lim_{n\to\infty} \frac{U_{n+1}}{U_n} > 1$, the series is not convergent.

3) find the values of x for which the series below is absolutely convergent

$$\frac{x}{27} + \frac{x^2}{125} + \cdots + \frac{x^n}{(2n+1)^3}$$

Solution

Applying D'Alembert Ratio

$$|U_n| = \frac{x^n}{(2n+1)^3}$$

$$|U_{n+1}| = \frac{x^{n+1}}{(2(n+1)+1)^3} = \frac{x^{n+1}}{(2n+2+1)^3} = \frac{x^{n+1}}{(2n+3)^3}$$

Ratio: $\frac{|U_{n+1}|}{|U_n|} = \frac{x^{n+1}}{(2n+3)^3} \times \frac{(2n+1)^3}{x^n}$

$$= \frac{x^n \cdot x^1}{(2n+3)^3} \times \frac{(2n+1)^3}{x^n}$$

$$= \frac{x(2n+1)^3}{(2n+3)^3}$$

$$\lim_{n\to\infty} \frac{|U_{n+1}|}{|U_n|} = \frac{x\left(\frac{2n}{n}+1/n\right)^3}{\left(\frac{2n}{n}+\frac{3}{n}\right)^3} = \frac{x\left(2+1/n\right)^3}{\left(2+3/n\right)^3}$$

$$= \frac{x(2+0)^3}{(2+0)^3} = \frac{\cancel{8}x}{\cancel{8}} = x$$

For absolute convergency, $\lim_{n\to\infty} \frac{|U_{n+1}|}{|U_n|} < 1$

$\therefore x < 1$ is the range of values for which the series is absolutely convergent

4) Evaluate using L'hopital's Rule

$$\lim_{x\to 0}\left[\frac{\sin x - \cos x}{x^3}\right]$$

<u>Solution</u>

$$\lim_{x\to 0}\left[\frac{\sin x - \cos x}{x^3}\right] = \frac{\sin 0 - \cos 0}{0^3} = \text{Indeterminate}$$

$$\lim_{x\to 0}\left[\frac{\cos x + \sin x}{3x^2}\right] = \frac{\cos 0 + \sin 0}{3(0)^2} = \text{Indeterminate}$$

$$\lim_{x \to 0} \left[\frac{-\sin x + \cos x}{6x} \right] = \frac{-\sin 0 + \cos 0}{6(0)} = \text{Indeterminate}$$

$$\lim_{x \to 0} \left[\frac{-\cos x - \sin x}{6} \right] = \frac{-\cos 0 - \sin 0}{6} = -\frac{1}{6} = \underline{\underline{-1/6}}$$