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16/ENG03/027
Civil Engineering
ENG 281

The parametric equation of a curve are as given in Equations (1) and (2)

$$x = \cos t + t\sin t$$
$$y = \sin t - t\cos t$$

In terms of t, determine:
i) an expression for the radius of curvature (R) and
ii) expressions for the coordinates (h, k) of the centre of curvature

Solution

$$x = \cos t + t\sin t$$
$$y = \sin t - t\cos t$$

$$\frac{dx}{dt}\cancel{\frac{dy}{dt}} = -\sin t + (t\cos t + \sin t(1))$$

$$\frac{dx}{dt}\cancel{\frac{dy}{dt}} = -\sin t + t\cos t + \sin t$$

$$\frac{dx}{dt} = t\cos t$$

$$\frac{dy}{dt} = \cos t + (-t\sin t + (1)\cos t)$$

$$\frac{dy}{dt} = \cos t - t\sin t + \cos t$$

$$\frac{dy}{dt} = 2\cos t - t\sin t$$

$$\therefore \frac{dy}{dx} = \frac{2\cos t - t\sin t}{t\cos t} = \frac{2\cos t}{t\cos t} - \frac{t\sin t}{t\cos t}$$

$$\frac{dy}{dx} = \frac{2}{t} - \frac{\sin t}{\cos t}$$

$$\frac{dy}{dx} = \frac{2}{t} - \tan t \quad ; \quad \frac{dy}{dx} = 2t^{-1} - \tan t$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}\frac{dt}{dt}\frac{dy}{dx} = \frac{d}{dt}\frac{dt}{dx}\frac{dy}{dx}$$

$$= \frac{d}{dt}\frac{dy}{dx}\frac{dt}{dx} = \frac{d}{dt}\left[\frac{dy}{x}\right]\frac{dt}{dx}$$

$$\frac{d}{dt}\left[\frac{dy}{dx}\right] = -2t^{-2} - \sec^2 t \quad ; \quad \frac{dt}{dx} = \frac{1}{t\cos t}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{-2t^{-2} - \sec^2 t}{t\cos t}$$

recall R, Radius of curvature =

$$R = \frac{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{3/2}}{\frac{d^2y}{dx^2}}$$

$$\therefore R = \left[\frac{\left[1 + \left(2t^{-1} - \tan t\right)^2\right]^{3/2}}{\frac{-2t^{-2} - \sec^2 t}{t\cos t}}\right]$$

(ii) Center of curvature (h, k)

recall, $h = x - R\sin\theta$

$k = y + R\cos\theta$

$$\tan\theta = \left[\frac{dy}{dx}\right]_P$$

$$\tan\theta = [2t^{-1} - \tan t]$$

$$\therefore \theta = \tan^{-1}[2t^{-1} - \tan t]$$

$$\therefore h = x - R\sin\theta$$

$$= (\cos t + t\sin t) - \left[\frac{[1 + (2t^{-1} - \tan t)^2]^{3/2}}{\frac{-2t^{-2} - \sec^2 t}{t\cos t}}\right] \sin[\tan^{-1}(2t^{-1} - \tan t)]$$

$$K = y + R\cos\theta$$

$$= (\sin t - t\cos t) + \left[\frac{[1 + (2t^{-1} - \tan t)^2]^{3/2}}{\frac{-2t^{-2} - \sec^2 t}{t\cos t}}\right] \cos[\tan^{-1}(2t^{-1} - \tan t)]$$