Name: ThankGod Winton
Dept: Mechanical Engineering
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(1) Evaluate the following limits of function.

(a) $\lim_{x \to \pi/2} \left\{ \frac{(x^2 - \pi/4)\sin(\cos x)}{x - \pi/2} \right\}$

solution

$\frac{dy}{dx} = \frac{(2x - 0)(-\sin^2 x)}{1 - 0}$, $y = \left\{ \frac{(x^2 - \pi/4)\sin(\cos x)}{x - \pi/2} \right\}$

Direct substitution $x \to \pi/2$

$= 2\left[(\pi/2)\right]\left(-\sin^2(\pi/2)\right]$

$= -\pi(-1)^2$

$= -\pi$

$\lim_{x \to \pi/2} \left[\frac{(x^2 - \pi/4)\sin(\cos x)}{x - \pi/2} \right] = -\pi$

(b) $\lim_{x \to 1/2} \ln \left\{ \exp \frac{(3x^2 + 2x - 1)}{x+1} \right\}$

solution

$\lim_{x \to -1} \ln \left[\exp \frac{(3x-1)(x+1)}{(x+1)} \right]$

$\lim_{x \to -1} \ln \left[\exp [3(-1) - 1] \right]$

$\ln (\exp(-3 - 1)] = -4$

(c) $\lim_{x \to 2+\sqrt{3}} \cos \left[\sin^{-1} \left(\frac{x - 2}{x - \sqrt{3}} \right) \right]$

$\cos \left[\sin^{-1} \left(\frac{\cancel{2} + \sqrt{3} - \cancel{2}}{2 + \cancel{\sqrt{3}} - \cancel{\sqrt{3}}} \right) \right] \Rightarrow \cos \left[\sin^{-1} \left(\frac{\sqrt{3}}{2} \right) \right]$

$\Rightarrow \cos 60 = 1/2$

d) $\lim_{x \to 4} \left[\frac{x^2 - 8x + 16}{x^2 - 5x + 4} \right]$

Solution

$\lim_{x \to 4} \left[\frac{\cancel{(x-4)}(x+4)}{(x-1)\cancel{(x-4)}} \right]$

$\lim_{x \to 4} \left[\frac{x+4}{x-1} \right]$

$= \frac{4-4}{4-1} = \frac{0}{3}$ (Undefined)

2) Determine whether each of the following series is convergent.

b) $\frac{2}{2\times3} + \frac{2}{3\times4} + \frac{2}{4\times5} + \frac{2}{5\times6} \; . \; . \; . \; . \; , \; \frac{2}{(n+1)(n+2)}$

Solution

$U_n = \frac{2}{(n+1)(n+2)}$, $U_{n+1} = \frac{2}{(n+2)(n+3)}$

$\lim_{n \to \infty} \frac{U_{n+1}}{U_n} = \frac{2}{(n+2)(n+3)} \times \frac{(n+1)(n+2)}{2}$

$\Rightarrow \frac{\cancel{2}(n^2+3n+2)}{\cancel{2}(n^2+5n+6)}$

$\frac{n^2/n^2 + 3n/n^2 + 2/n^2}{n^2/n^2 + 5n/n^2 + 6/n^2} \Rightarrow \frac{1 + 3/n + 2/n^2}{1 + 5/n + 6/n^2}$

$n \to \infty$, $\frac{1 + 3/0 + 2/0}{1 + 5/0 + 6/0} = \frac{1}{1} = \underline{\underline{1}}$

$\therefore \frac{U_{n+1}}{U_n} = 1$, series is divergent or convergent.

Further test,

$\lim_{n \to \infty} U_n = \frac{2}{(n+1)(n+2)} \Rightarrow \frac{2}{(0+1)(0+2)} = \frac{2}{2} = 1$

$U_n \neq 0 \therefore$ the series is divergent.

6) $\frac{2}{1^2} + \frac{2^2}{2^2} + \frac{2^3}{3^2} + \frac{2^4}{4^2} + \ldots \frac{2^n}{n^2}$

solution.

$U_n = \frac{2^n}{n^2}$, $U_{n+1} = \frac{2^{n+1}}{(n+1)^2}$

$\lim_{n\to\infty} = \frac{U_{n+1}}{U_n} = \frac{2^{n+1}}{n^2+2n+1} \times \frac{n^2}{2^n} \Rightarrow \frac{\cancel{2^n} \times 2 \times n^2}{\cancel{2^n}(n^2+2n+1)}$

$\lim_{n\to\infty} \frac{2n^2}{n^2+2n+1} = \frac{2n^2/n^2}{n^2/n^2 + 2n/n^2 + 1/n^2} \Rightarrow \frac{2}{1+2/n+1/n^2}$

$n \to \infty$, $\frac{2}{1+0+0} = 2.$

$\therefore \frac{U_{n+1}}{U_n} > 1$, the series is divergent.

6c) $U_n = \frac{1+2n^2}{1+n^2}$

$\lim_{n\to\infty} \frac{1+2n^2}{1+n^2} = \frac{1/n^2+2}{1/n^2+1} = \frac{2+0}{1+0} = \frac{2}{1} = 2$

$n \to 1/n \to 0$

$U_n \neq 0 \therefore$ series is divergent

3) Find the range of values of x for which the series below is absolutely convergent,

$\frac{x}{27} + \frac{x^2}{125} + \ldots + \frac{x^n}{(2n+1)^3}$

solution.

$U_n = \frac{x^n}{(2n+1)^3}$, $U_{n+1} = \frac{x^{n+1}}{(2n+2)^3}$

$\lim \frac{U_{n+1}}{U_n} = \frac{x^{n+1}}{(2n+2)^3} \times \frac{(2n+1)^3}{x^n}$

$= \frac{x(2n+1)^3}{(2n+2)^3} \Rightarrow \frac{x[8n^3+12n^2+n+1]}{8n^3+24n^2+24n+8}$

$$= \frac{x\left(8n^2/n^3 + 12n^2/n^3 + n/n^3 + 1/n^3\right)}{\left(8n^3/n^3 + 24n^2/n^3 + 24n/n^3 + 8/n^3\right)}$$

$$\Rightarrow \frac{x\left(8 + 12/n + 1/n^2 + 1/n^3\right)}{\left(8 + 12/n + 24/n^2 + 8/n^3\right)}$$

as $n \to \infty$, $1/n \to 0$

$$\frac{x(8+0+0+0)}{(8+0+0+0)} = \frac{8x}{8} \Rightarrow x = 1$$

$\therefore x \leq 1$.

(4) Evaluate using L'Hopital Rule

$$\lim_{x \to 0}\left(\frac{\sin x - \cos x}{x^3}\right)$$

Solution:

By using L'Hopitals rule

$$y = \left[\frac{\sin x - \cos x}{x^3}\right]$$

$$\frac{dy}{dx} = \left[\frac{\cos x + \sin x}{3x^2}\right]$$

$$\frac{d^2y}{dx^2} = \frac{-\sin x + \cos x}{6x}$$

$$\frac{d^3y}{dx^3} = \frac{-\cos x - \sin x}{6}$$

$$\lim_{x \to 0} = \frac{-\cos 0 - \sin 0}{6} = \cancel{\pm 0}\ \frac{-1-0}{6} = \frac{-1}{6}$$

$$\lim_{x \to 0}\left[\frac{\sin x - \cos x}{x^3}\right] = \frac{-1}{6}$$