16/ENG06/004

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The parametric equations of a curve are as given in equations (1) and (2):

$x = \cos t + t \sin t$ ——— (1)

$y = \sin t - t \cos t$ ——— (2)

In terms of t, determine:

(i) An expression for the radius of curvature (R)

(ii) Expressions for the co-ordinates (h, k) of the centre of curvature.

SOLUTION

Recall,

$$R = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

(i) $x = \cos t + t \sin t$

$$\frac{dx}{dt} = -\sin t + t\cos t + \sin t \;(1)$$

$$\frac{dx}{dt} = t\cos t$$

$y = \sin t - t\cos t$

$$\frac{dy}{dt} = \cos t + t\sin t - \cos t \;(1)$$

$$\frac{dy}{dt} = t\sin t$$

Using Quotient rule

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= \cancel{t}\sin t \times \frac{1}{\cancel{t}\cos t}$$

$$\frac{dy}{dx} = \frac{\sin t}{\cos t}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{\sin t}{\cos t}\right) \cdot \frac{dt}{dx}$$

Let $u = \sin t$

$v = \cos t$

$$\frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$u = \sin t \quad \frac{du}{dt} = \cos t$

$v = \cos t \quad \frac{dv}{dt} = -\sin t$

$$\frac{d^2y}{dx^2} = \frac{\cos t - (-\sin^2 t)}{\cos^2} \times \frac{dt}{dx}$$

$$= \frac{\cos t + \sin^2 t}{\cos^2 t} \times \frac{1}{t\cos t}$$

Recall that $\sin^2\theta + \cos^2\theta = 1$

$$\therefore \frac{d^2y}{dx^2} = \frac{1}{\cos^2 t} \times \frac{1}{t\cos t}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{1}{t\cos^3 t}$$

$$R = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$R = \frac{\left[1 + \left(\frac{\sin t}{\cos t}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}, \qquad R = \left[1 + \frac{\sin^2 t}{\cos^2 t}\right]^{3/2} \times \frac{t\cos^3 t}{1}$$

$$R = \left[\frac{\cos^2 t + \sin^2 t}{\cos^2 t}\right]^{3/2} \times t\cos^3 t$$

$$= \frac{1^{3/2}}{(\cos^2 t)^{3/2}} \times t\cos^3 t$$

$$R = \frac{1}{\cancel{\cos^3 t}} \times t\cancel{\cos^3 t}$$

$\therefore R = t$ ∴ The expression for the radius of curvature (R) is t

ii) Coordinates (h, k)

$h = x_1 - R\sin\theta$ — — — (1)

$k = y_1 + R\cos\theta$ — — — (2)

From the previous question, we have $R = t$; $\theta = t$

$x_1 = \cos t + t\sin t$

$y_1 = \sin t - t\cos t$

Substituting x_1 y_1 θ and R into equation (1) and (2)

i.e $h = (\cos t + t\sin t) - t\sin(t)$ —— (3)

$k = (\sin t - t\cos t) + t\cos(t)$ —— (4)

Solving eqn (3)

$h = (\cos t + t\sin t) - t\sin(t)$

$\therefore h = \cos t$

Solving equation 4

$k = (\sin t - t\cos t) + t\cos(t)$

$k = \sin t$

$\therefore (h, k) = (\cos t, \sin t)$

The expressions for the coordinates (h, k) of the centre of curvature is $(\cos t, \sin t)$.