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1. Evaluate the following limits of function.

(a) $\lim_{x \to \pi/2} \left\{ \frac{(x^2 - \pi/4)\sin(\cos x)}{x - x/2} \right\}$ (b) $\lim_{x \to \pi/2} \ln \left[\frac{\exp(3x^2 + 2x - 1)}{x + 1} \right]$

(c) $\lim_{x \to 2\sqrt{3}} \cos \left\{ \frac{\sin^{-1}(x-2)}{(x - \sqrt{3})} \right\}$ (d) $\lim_{x \to 4} \left\{ \frac{x^2 - 3x + 16}{x^2 - 5x + 4} \right\}$

2. Determine whether each of the following series is convergent.

(a) $\frac{2}{2\times 3} + \frac{2}{3\times 4} + \frac{2}{4\times 5} + \frac{2}{5\times 6} + \cdots$

(b) $\frac{2}{1^2} + \frac{2}{2^2} + \frac{2}{3^2} + \frac{2}{4^2} + \cdots$

(c) $U_n = \frac{1 + 2n^2}{1 + n^2}$

3. Find the range of values of x for which the series below is absolutely convergent;

$\frac{x}{27} + \frac{x^2}{125} + \cdots + \frac{x^n}{(2n+1)^3}$

4. Evaluate using L'Hopital Rule.

$\lim_{x \to 0} \left(\frac{\sin x - \cos x}{x^3} \right)$

Soln

1. (a) $\lim_{x \to \pi/2} \left\{ \frac{(x^2 - \pi/4)\sin(\cos x)}{x - \pi/2} \right\}$

$\frac{dy}{dx} = \frac{(2x-0)(-\sin^2 x)}{1-0}$, $y = \left[\frac{(x^2 - \pi/4)\sin(\cos x)}{x - \pi/2} \right]$

Direct substitng $x \to \pi/2$

$= 2\left[\pi/2 (-\sin^2(\pi/2)\right]$

$= -\pi(-1)^2$

$= -\pi$

$\lim_{x \to \pi/2} \left[\frac{(\pi^2 - \pi/4)\sin(\cos x)}{x - \pi/2} \right] = -\pi$

(b) $\lim_{x \to -1} \ln\left[\exp\left(\frac{3x^2 + 2x - 1}{x+1}\right)\right]$

$\ln\left[\exp\frac{(3x-1)(x+1)}{\cancel{x+1}}\right]$

$\lim_{x \to -1} \ln\left[\exp[3(-1)-1)\right]$

$\ln(\exp(-3-1)] = -4$

(c) $\lim_{x \to 2+\sqrt{3}} \cos\left[\sin^{-1}\left(\frac{x-2}{x-\sqrt{3}}\right)\right]$

$\cos\left[\sin^{-1}\left(\frac{2+\sqrt{3}-2}{2+\sqrt{3}-\sqrt{3}}\right)\right] \Rightarrow \cos\left[\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right]$

$\Rightarrow \cos 60 = 1/2$

d) $\lim_{x \to 4} \left[\frac{x^2 - 8x + 16}{x^2 - 5x + 4}\right]$

$\lim_{x \to 4} \left[\frac{(x-4)(x+4)}{(x-1)(x-4)}\right]$

$$\lim_{x \to 4} \left[\frac{x+4}{x-1}\right]$$

$$= \frac{4-4}{4-1} = \frac{0}{3} \text{ (undefined)}$$

2 ⓐ $\frac{2}{2\times3} + \frac{2}{3\times4} + \frac{2}{4\times5} + \frac{2}{5\times6} \cdots \frac{2}{(n+1)(n+2)}$

$$U_n = \frac{2}{(n+1)(n+2)}, \quad U_{n+1} = \frac{2}{(n+2)(n+3)}$$

$$\lim_{n \to \infty} \frac{U_{n+1}}{U_n} = \frac{2}{(n+2)(n+3)} \times \frac{(n+1)(n+2)}{2}$$

$$= \frac{\cancel{2}(n^2+3n+2)}{\cancel{2}(n^2+5n+6)}$$

$$\frac{n^2/n^2 + 3n/n^2 + 2/n^2}{n^2/n^2 + 5n/n^2 + 6/n^2} \Rightarrow \frac{1 + 3/n + 2/n^2}{1 + 5/n + 6/n^2}$$

$$n \to \infty, \quad \frac{1 + 3/0 + 2/0}{1 + 5/0 + 6/0} = \frac{1}{1} = 1,$$

$\therefore \frac{U_n+1}{U_n} = 1$, series is divergent or convergent.

Further test.

$$\lim_{n \to \infty} U_n = \frac{2}{(n+1)(n+2)} \Rightarrow \frac{2}{(0+1)(0+2)} = \frac{2}{2} = 1,$$

$U_n \neq 0 \therefore$ the series is divergent.

ⓑ $\frac{2}{1^2} + \frac{2^2}{2^2} + \frac{2^3}{3^2} + \frac{2^4}{4^2} + \cdots \frac{2^n}{n^2}$

$$U_n = \frac{2^n}{n^2}, \quad U_{n+1} = \frac{2^{n+1}}{(n+1)^2}$$

$$\lim_{n \to \infty} = \frac{U_n+1}{U_n} = \frac{2^{n+1}}{n^2+2n+1} \times \frac{n^2}{2^n} \to \frac{2^n \times 2 \times n^2}{2^n(n^2+2n+1)}$$

$$\lim_{n\to\infty} = \frac{2n^2/n^2}{n^2/n^2 + 2n/n^2 + 1/n^2} \Rightarrow \frac{2}{1 + 2/n + 1/n^2}$$

$n \to \infty$, $\frac{2}{1+0+0} = 2$

$\therefore U_{n+1}/U_n > 1$, the series is divergent.

(c) $U_n = 1+2n^2/1+n^2$

$$\lim_{n\to\infty} \frac{1+2n^2}{1+n^2} = \frac{1/n^2 + 2}{1/n^2+1} = \frac{2+0}{1+0} = \frac{2}{1} = 2$$

$n \to 1/n \to 0$, $U_n \neq 0$, $\therefore$ Series is divergent.

3 $U_n = x^n/(2n+1)^3$, $U_{n+1} = x^{n+1}/(2n+1)^3$

$$\lim U_{n+1}/U_n = x^{n+1}/(2n+2)^3 \times (2n+1)^3/x^n$$

$$= \frac{x(2n+1)^3}{(2n+2)^3} \Rightarrow \frac{x[8n^3 + 12n^2 + n + 1]}{8n^3 + 24n^2 + 24n + 8}$$

$$= \frac{x(8n^3/n^3 + 12n^2/n^3 + n/n^3 + 1/n^3)}{(8n^3/n^3 + 24n^2/n^3 + 24n/n^3 + 8/n^3)} = \frac{x(8 + 12/n + 1/n^2 + 1/n^3)}{(8 + 12/n + 12/n^2 + 8/n^3)}$$

as $n \to \infty$, $1/n \to 0$

$$\frac{x(8+0+0+0)}{(8+0+0+0)} = \frac{8x}{8} \Rightarrow x = 1 \quad \therefore x \leq 1$$

4. By L'hopital's rule.

$$y = \left[\frac{\sin x - \cos x}{x^3}\right]$$

$$dy/dx = \left[\frac{\cos x + \sin x}{3x^2}\right]$$

$$\frac{d^2y}{dx^2} = \frac{-\sin x + \cos x}{6x}, \quad \frac{d^3y}{dx^3} = \frac{-\cos x - \sin x}{6}$$

$$\lim_{x\to 0} = \frac{-\cos 0 - \sin 0}{6} = \frac{-1-0}{6} = -1/6$$

$$\lim_{x\to 0}\left[\frac{\sin x - \cos x}{x^3}\right] = -1/6$$