ADIOTOMRE WILLIAM
PETROLEUM ENGINEERING
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1 Evaluate the following limits of functions

a $\lim_{x \to \frac{\pi}{2}} \left(\frac{\left(x^2 - \frac{\pi}{4}\right) \sin(\cos x)}{x - \frac{\pi}{2}} \right)$

let $u = x^2 - \frac{\pi}{4}$

$v = \sin(\cos x)$

$\frac{dy}{dx} = 2x$

let $y = \sin(\cos x)$ let $a = \cos x$ $\therefore y = \sin a$

$\frac{dy}{da} = \cos a$ $\frac{da}{dx} = -\sin a$

$\frac{dy}{dx} = \frac{dy}{da} \times \frac{da}{dx} = \cos a \times (-\sin a)$

$= [\cos(\cos x)] \times -\sin x$

$\frac{dv}{dx} = -\sin x \cos(\cos x)$

$\frac{dy}{dx} = \frac{v\,du}{dx} + \frac{u\,dv}{dx}$

$= \sin(\cos x)(2x) + x^2 - \frac{\pi}{4}[-\sin x \cos(\cos x)]$

denomenator

$\frac{dy}{dx} = 1$

$$\lim_{x \to \frac{\pi}{2}} = \frac{2x \sin(\cos x) + \left(x^2 - \frac{\pi}{4}\right)(-\sin x \cos(\cos x))}{1}$$

$$= \frac{2\left(\frac{\pi}{2}\right)\sin\left(\cos\frac{\pi}{2}\right) + \left(\frac{\pi}{2}\right)^2 - \frac{\pi}{4}\left(-\sin\left(\frac{\pi}{2}\right)\cos\left(\cos\left(\frac{\pi}{2}\right)\right)\right)}{1}$$

$$= \pi(0) + \frac{\pi^2}{4} - \frac{\pi}{4}(-1 \times 1)$$

$$= -\frac{\pi^2}{4} + \frac{\pi}{4}$$

$$\therefore \lim_{x \to \frac{\pi}{2}} \left[\frac{\left(x^2 - \frac{\pi}{4}\right)\sin(\cos x)}{x - \frac{\pi}{2}}\right] = \frac{\pi(1 - x)}{4}$$

b $$\lim_{x \to \frac{\pi}{2}} \ln\left(\exp\frac{(3x^2 + 2x - 1)}{x + 1}\right)$$

$$\frac{3x^2 + 2x - 1}{x + 1}$$

$$\frac{6x + 2}{1}$$

$$\lim_{x \to \frac{\pi}{2}} = \frac{6\frac{\pi}{2} + 2}{1} = 3\pi + 2$$

c $\lim_{x \to 2+\sqrt{3}} \cos\left(\sin^{-1}\frac{(x-2)}{x-\sqrt{3}}\right)$

$\lim_{x \to 2+\sqrt{3}} \cos\left(\sin^{-1}\left(\frac{2+\sqrt{3}-2}{2+\sqrt{3}-\sqrt{3}}\right)\right)$

$\cos\left(\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)$

$\cos\left(\sin^{-1}(60)\right)$

$\cos 60$

$= \frac{1}{2}$

$\therefore \lim_{x \to 2+\sqrt{3}} = \frac{1}{2}$

d $\lim_{x \to 4}\left(\frac{x^2-8x+16}{x^2-5x+4}\right)$

$\lim_{x \to 4}\left(\frac{2x-8}{2x-5}\right)$

$\left(\frac{2(4)-8}{2(4)-5}\right) = \frac{0}{3} = 0$

$\lim_{x \to 4}\left(\frac{x^2-8x+16}{x^2-5x-5}\right) = 0$

2 Determine whether each of the following Series is Convergent

a $\frac{2}{2\times3} + \frac{2}{3\times4} + \frac{2}{4\times5} + \frac{2}{5\times6} + \ldots$

$$U_n = \frac{2}{(n+1)(n+2)}$$

$$U_{n+1} = \frac{2}{(n+1+1)(n+1+2)}$$

$$= \frac{2}{(n+2)(n+3)}$$

$$\frac{U_{n+1}}{U_n} = \frac{\frac{2}{(n+1)(n+2)}}{\frac{2}{(n+2)(n+3)}} = \frac{2}{(n+1)(n+2)} \times \frac{(n+2)(n+3)}{2}$$

$$= \frac{(n+2)(n+3)}{(n+1)(n+2)} = \frac{n^2+3n+2n+6}{n^2+2n+1n+2} = \frac{n^2+5n+6}{n^2+3n+2}$$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \frac{\frac{n^2}{n^2}+\frac{5n}{n^2}+\frac{6}{n^2}}{\frac{n^2}{n^2}+\frac{3n}{n^2}+\frac{2}{n^2}} = \frac{1+\frac{5}{n}+\frac{6}{n^2}}{1+\frac{3}{n}+\frac{2}{n^2}}$$

$$= \frac{1+0+0}{1+0+0}$$

$$= 1 \text{ (inconclusive)}$$

Using the P-Series to compare with the Series

$$U_n = \frac{2}{(n+1)(n+2)}$$

using the P-Series to Compare with the Series

$$U_n = \frac{x}{(n+1)(n+2)} \qquad \text{P-Series} = \frac{1}{n^p}$$

for $p=2$

$$\frac{x}{(n+1)(n+2)} > \frac{1}{n^2}$$

Test for $n = 1,2,3,4$

Note $x=2$ for $u_n = \frac{x}{(n+1)(n+2)}$

$$u_n = \frac{2}{(n+1)(n+2)}$$

for $n=1$

$$\frac{2}{2\times 3} < \frac{1}{1}$$

for $n=2$

$$\frac{2}{3\times 4} < \frac{1}{4}$$

for $n=3$

$$\frac{2}{6\times 7} > \frac{1}{25}$$

for $n=4$

$$\frac{2}{5\times 6} > \frac{1}{16}$$

$\therefore$ therefore the series $u_n < \frac{1}{n^2}$ for only $n \leq 3$

therfore the series $u_n = \frac{x}{(n+1)(n+2)}$ cannot be said to be a converging series. It may be divergent

b $\frac{2}{1^2} + \frac{2}{2^2} + \frac{2}{3^2} + \frac{2}{4^2}$

$\therefore u_n = \frac{x}{n^2}$, $u_{n+1} = \frac{x}{(n+1)^2}$

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \frac{x}{n^2+2n+1} \times \frac{n^2}{x} = \frac{n^2}{n^2+2n+1}$$

dividing through by n^2

$$\frac{u_{n+1}}{u_n} = \frac{1}{1+\frac{2}{n}+\frac{1}{n^2}} = 1 \text{ (inconclusive)}$$

Comparison with the p series

$U_n = \frac{2}{n^2}$ and $\frac{1}{n^2}$ for $p=2$

Since $x = 2$ for U_n

$U_n > \frac{1}{n^p}$ (ie $\frac{2}{n^2} > \frac{1}{n^2}$)

Therefore the series $U_n = \frac{x}{n^2}$ does not converges but diverges

2c $U_n = \frac{1+2n^2}{1+n^2}$

$U_{n+1} = \frac{1+2(n+1)^2}{n^2+2n+2} \cdot \frac{1+n^2}{1+2n^2}$

divide through n^2

$$\lim_{n \to \infty} = \frac{\left(2 + \frac{4}{n} + \frac{2}{n^2}\right) \cdot \left(\frac{1}{n^2} + 1\right)}{\left(1 + \frac{2}{n} + \frac{2}{n^2}\right)\left(\frac{1}{n^2} + 2\right)} = \frac{2}{2} = 1$$

$\therefore U_n = \frac{1+2n^2}{1+n^2}$ is inconclusive

Comparison with the p-series

$U_n = \frac{1+2n^2}{1+n^2}$ P-series $= \frac{1}{n^2}$ for $P=2$

$\therefore$ ~~The~~ S for $n=1$

$U_n = \frac{3}{2} > 1$

for $n=2$

$\frac{9}{5} = \frac{1}{4}$

from the above test $U_n > \frac{1}{n^p}$

i.e $\frac{1+2n^2}{1+n^2} > \frac{1}{n^2}$ for $P=2$

$\therefore$ The Series $U_n + \frac{1+2n^2}{1+n^2}$ diverges it doesnt converge

3 Find the range of values for x for which the series below is absolutly Convergent

$$\frac{x}{27} + \frac{x^2}{125} + \quad \frac{+x^2}{(2n+1)^3}$$

$$|U_n| = \frac{x^n}{(2n+1)^3}$$

$$|U_{n+1}| = \frac{x^{n+1}}{(2(n+1)+1)^3} = \frac{x^{n+1}}{(2n+3)^3}$$

$$\left|\frac{U_{n+1}}{U_n}\right| = \left|\frac{x^{n+1}}{(2n+3)^3} \div \frac{x^n}{(2n+1)^3}\right| = \left|\frac{x^n \cdot x^1}{(2n+3)^3} \times \frac{(2n+1)^3}{\cancel{x^n}}\right|$$

$$= \frac{x(2n+1)^3}{(2n+3)^3}$$

$$\lim_{x \to \infty} \frac{x(2n+1)^3}{(2n+3)^3}$$

$$\frac{x(2n+1)(2n+1)(2n+1)}{(2n+3)(2n+3)(2n+3)}$$

$$\frac{x(4n^2+4n+1)(2n+1)}{(4n^2+12n+9)(2n+3)}$$

$$\frac{x(8n^3+8n^2+2n+4n^2+4n+1)}{8n^3+24n^2+18n+12n^2+36n+27}$$

$$\frac{x(8n^3+12n^2+6n+1)}{8n^3+36n^2+54n+27}$$

$$\lim_{x-\infty} \frac{x(2n+1)^3}{(2n+3)^3} = \frac{\frac{8n^3}{n^3} + \frac{12n^2}{n^3} + \frac{6n}{n^3} + \frac{1}{n^3}}{\frac{8n^3}{n^3} + \frac{36n^2}{n^3} + \frac{54n^3}{n^3} + \frac{27}{n^3}}$$

$$= \frac{8 + \frac{12}{n} + \frac{6}{n^2} + \frac{1}{n^3}}{8 + \frac{36}{n} + \frac{54}{n^2} + \frac{27}{n^3}}$$

$$= \frac{8 + 0 + 0 + 0}{8 + 0 + 0 + 0}$$

$$= 1$$

Since the $\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = 1$

The series Converges $\lim_{n \to \infty} \frac{u_{n+1}}{u_n} < 1$

$\therefore$ the range of values for which the series is Convergent is $x < 1$

4 Evaluate using L'Hopital's Rule

$$\lim_{x \to 0} \left(\frac{\sin x - \cos x}{x^3} \right)$$

$$\left(\frac{\cos x + \sin x}{3x^2} \right)$$

$$= \left(\frac{-\sin x + \cos x}{6x} \right)$$

$$= \left(\frac{-\cos x - \sin x}{6} \right) = \frac{-1 - 0}{6} = \frac{-1}{6}$$

$$= -0.167$$