ORUMWENSE OSAGUIYI ERNEST
16/ENG06/061
MECHANICAL
ENG 281

The parametric equations of a Curve are as given as equations ① and ②

$$x = \cos t + t \sin t$$
$$y = \sin t - t \cos t$$

In terms of t determine:

(i) An expression for the radius of Curvature (R); and
(ii) Expressions for the Co-ordinates (h, k) of the Centre of Curvature

Soln

Recall

i) $$R = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$x = \cos t + t \sin t$$
$$y = \sin t - t \cos t$$

$$\frac{dx}{dt} = -\sin t + \sin t(1) + t(\cos t)$$
$$= -\sin t + \sin t + t \cos t$$
$$\therefore \frac{dx}{dt} = t\cos t$$

$$\frac{dy}{dt} = \cos t - t(-\sin t) - \cos t(1)$$
$$= \cos t + t \sin t - \cos t$$
$$\frac{dy}{dt} = t \sin t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{t\sin t}{1} \times \frac{1}{t\cos t} = \frac{\cancel{t}\sin t}{\cancel{t}\cos t} = \frac{\sin t}{\cos t}$$

$\frac{d^2y}{dx^2}$, by product rule $\dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$; $u = \sin t$, $v = \cos t$

Case $\frac{du}{dx} = \frac{du}{dt} \div \frac{dx}{dt} = \frac{du}{dt} \times \frac{dt}{dx}$

$\frac{du}{dt} = \cos t$; $\frac{dx}{dt} = t\cos t$ $\therefore \frac{du}{dx} = \frac{\cos t}{t\cos t}$

$\frac{dv}{dx} = \frac{dv}{dt} \div \frac{dx}{dt} = \frac{dv}{dt} \times \frac{dt}{dx}$

$\frac{dv}{dt} = -\sin t$; $\frac{dx}{dt} = t\cos t$ $\therefore \frac{dv}{dx} = \frac{-\sin t}{t\cos t}$

$$= \frac{\cos t\left(\frac{\cos t}{t\cos t}\right) - \sin t\left(\frac{-\sin t}{t\cos t}\right)}{\cos^2 t} = \frac{\frac{\cos t}{t} + \frac{\sin^2 t}{t\cos t}}{\cos^2 t}$$

$$= \frac{\cos^2 t + \sin^2 t}{t\cos t} \times \frac{1}{\cos^2 t} =$$

Recall $\sin^2\theta + \cos^2\theta = 1$

$$= \frac{1}{t\cos t} \times \frac{1}{\cos^2 t} = \frac{1}{t\cos^3 t} \quad \therefore \frac{d^2y}{dx^2} = \frac{1}{t\cos^3 t}$$

$$R = \frac{\left[1 + \left(\frac{\sin t}{\cos t}\right)^2\right]^{3/2}}{1} \div \frac{1}{t\cos^3 t}$$

$$R = 1 + \cancel{\sin^2 t}\left[\frac{\cos^2 t + \sin^2 t}{\cos^2 t}\right]^{3/2} \times t\cos^3 t$$

$$R = \frac{1}{(\cos^2 t)^{3/2}} \times t\cos^3 t \quad ; \quad R = \frac{1}{\cos^3 t} \times t\cos^3 t$$

$R = t$ $\therefore$ Radius of Curvature $= t$

ii) (h, k)

Recall, $h = x_1 - R\sin\theta$ - - - (1)

$k = y_1 + R\cos\theta$ - - - (2)

$R = t$; $\theta = t$

$x_1 = \cos t + t\sin t$

$y_1 = \sin t - t\cos t$

$h = \cos t + t\sin t - t\sin t$

$h = \cos t$,

$k = \sin t - t\cos t + t\cos t$

$k = \sin t$ //

The expressions for the Co-ordinates (h, k) of the Centre of Curvature is $(\cos t, \sin t)$.