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Engineering Thermodynamics

The parametric equations of a curve are as given in equations (1) and (2)

$$x = \cos t + t \sin t \quad \text{--- (1)}$$

$$y = \sin t - t \cos t$$

In terms of t determine:

- i) An expression for the radius of curvature (R)
- ii) Expressions for the coordinates (h, k) of the centre of curvature

Solution

$$x = \cos t + t \sin t$$

$$y = \sin t - t \cos t$$

to find $\frac{dx}{dt}$

$$\text{let } u = \cos t \quad \text{and } v = t \sin t$$

$$\therefore \frac{dx}{dt} = \frac{du}{dt} + \frac{dv}{dt}$$

but to find $\frac{dv}{dt}$

Using product rule

$$\text{let } u = t \quad \text{and } v = \sin t$$

$$\therefore \frac{du}{dt} = 1 \quad \text{and} \quad \frac{dv}{dt} = \cos t$$

$$\therefore \frac{dv}{dt} = u \frac{dv}{dt} + v \frac{du}{dt}$$

$$\frac{dv}{dt} = t(\cos t) + \sin t(1)$$

$\frac{dv}{dt}$

$$\frac{dv}{dt} = \sin t + t \cos t$$

$\frac{dv}{dt}$

$$\therefore \frac{dx}{dt} = -\sin t + \sin t + t \cos t = t \cos t$$

$\frac{dx}{dt}$

Now

$$\frac{dy}{dt}$$

Also let $u = \sin t$ and $v = -t \cos t$

$$\frac{du}{dt} = \cos t$$

but $\frac{dv}{dt}$

let $w = -t$ and $m = \cos t$

$$\therefore \frac{dw}{dt} = -1 \quad \frac{dm}{dt} = -\sin t$$

$$\therefore \frac{dv}{dt} = w \frac{dm}{dt} + m \frac{dw}{dt}$$

$$\frac{dv}{dt} = -t(-\sin t) + \cos t(-1)$$

$$\frac{dv}{dt} = t \sin t - \cos t$$

$$\therefore \frac{dy}{dt} = \cos t + t \sin t - \cos t$$

$$\frac{dy}{dt} = t \sin t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{t \sin t}{t \cos t} = \frac{\sin t}{\cos t}$$

Now

$$\frac{d^2y}{dx^2}$$

$$dx^2$$

Using quotient rule

$$\text{let } \frac{\sin t}{\cos t} = \frac{u}{v}$$

$$\cos t \quad v$$

Recall

$$\frac{d^2y}{dx^2} = \frac{v \frac{du}{dt} - u \frac{dv}{dt}}{v^2}$$

$$\text{Since } u = \sin t \\ v = \cos t$$

$$\frac{du}{dt} = \cos t \\ \frac{dv}{dt} = -\sin t$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dt} \left(\frac{dy}{dx} \right) \frac{dt}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{\cos t (\cos t) - [\sin t \times \sin t]}{\cos^2 t} \frac{dt}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{\cos^2 t + \sin^2 t}{\cos^2 t} \frac{dt}{dx}$$

$$\frac{d^2y}{dx^2} = \text{Recall } \cos^2 t + \sin^2 t = 1$$

$$\therefore \frac{d^2y}{dx^2} = \frac{1}{\cos^2 t} \frac{dt}{dx}$$

$$\text{Recall } \frac{dt}{dx} = \frac{1}{t \cos^2 t}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{1}{\cos^2 t} \times \frac{1}{t \cos^2 t} = \frac{1}{t \cos^4 t}$$

Recall

$$R = \frac{1}{\sqrt{1 + \left(\frac{d^2y}{dx^2} \right)^2}}$$

$$R = \frac{1}{\left[1 + \left(\frac{d^2y}{dx^2} \right)^2 \right]^{3/2}}$$

$$\therefore R = \frac{1}{\left[1 + \left(\frac{d^2y}{dx^2} \right)^2 \right]^{3/2}}$$

$$R = \frac{1}{\left[1 + \left(\frac{\sin t}{\cos t} \right)^2 \right]^{3/2}} = \frac{1}{\cos^3 t}$$

$$R = \frac{1 + \sin^2 t}{\cos^3 t} = \frac{1}{\cos^3 t}$$

$$R = \frac{\cos^2 t + \sin^2 t}{\cos^3 t} \times t \cos^3 t$$

$$R = \frac{1}{\cos^3 t}$$

$$R = \frac{1}{\left(\sqrt{\cos^2 t} \right)^3} \times t \cos^3 t$$

$$R = \frac{1}{\cos^3 t} \times t \cos^3 t$$

$$R = t$$

ii) (h, k)

Recall, $h = x_1 - R \sin \theta$ --- (1)

$k = y_1 + R \cos \theta$ --- (2)

$R = t$ $\theta = t$

$x_1 = t \cos t + t \sin t$

$y_1 = t \sin t - t \cos t$

Substituting R, x_1, y_1 into h and k [i.e. equ (1) and (2)]

$h = t \cos t + t \sin t - t \sin t$

$h = t \cos t$

$k = t \sin t - t \cos t + t \cos t$

$k = t \sin t$

Therefore the co-ordinates for the centre can be given as
 $(t \cos t, t \sin t)$