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16/ENG04/050
Elect/ Elect
ENG 281 Assignment

The parametric equations of a curve are as given in equations 1 and 2

$x = \cos t + t\sin t$

$y = \sin t - t\cos t$

In terms of t, determine:

i an expression for the radius of curvature (R), and

ii expressions for the coordinates (h, k) of the centre of curvature

Solution

$x = \cos t + t\sin t$

$y = \sin t - t\cos t$

$\frac{dx}{dt} = -\sin t + (t\cos t + \sin t(1))$

$\frac{dx}{dt} = -\sin t + t\cos t + \sin t$

$\frac{dx}{dt} = t\cos t$

$\frac{dy}{dt} = \cos t - (-t\sin t + \cos t(1))$

$\therefore \frac{dy}{dt} = t\sin t$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

$$= \frac{t\sin t}{t\cos t} = \frac{\sin t}{\cos t}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{\sin t}{\cos t}\right) \cdot \frac{dt}{dt}$$

$$= \frac{v\frac{dy}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{v\frac{du}{dt} - u\frac{dv}{dt}}{v^2}$$

where $u = \sin t$, $v = \cos t$

$\frac{dy}{dt} = \cos t$, $\frac{dv}{dt} = -\sin t$

$$\frac{d^2y}{dx^2} = \frac{\cos t(\cos t) - \sin t(-\sin t)}{\cos^2 t}$$

$$\cancel{\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\sin\right.}$$

$$\frac{d^2y}{dx^2} = \frac{\cos^2 t + \sin^2 t}{\cos^2 t}$$

Recall that $\cos^2 t + \sin^2 t = 1$

$$\therefore \frac{d^2y}{dx^2} = \frac{1}{\cos^2 t} \times \frac{dt}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{1}{\cos^2 t} \times \frac{1}{t\cos t} = \frac{1}{t\cos^2 t}$$

i Recall $R = \dfrac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$

$$R = \left[1 + \left(\frac{\sin t}{\cos t}\right)^2\right]^{3/2} \div \frac{d^2y}{dx^2}$$

$$R = \left[1 + \frac{\sin^2 t}{\cos^2 t}\right]^{3/2} \times t\cos^3 t$$

$$R = \left[\frac{\cos^2 t + \sin^2 t}{\cos^2 t}\right]^{3/2} \times t\cos^3 t$$

$$R = \left(\frac{1}{\cos^2 t}\right)^{3/2} \times t\cos^3 t$$

$$= \frac{1}{(\cos t)^{2\times 3/2}} \times t\cos^3 t$$

$$= \frac{1}{\cancel{\cos^3 t}} \times t\,\cancel{\cos^3 t}$$

$$= 1 \times t = t$$

$$\therefore R = t$$

ii For the coordinate (h, k) of the centre of curvature

Recall $h = x - R\sin\theta$

$k = y + R\cos\theta$

where $R = t$

$\theta = \tan^{-1}\left[\frac{dy}{dx}\right]$

$= \tan^{-1}(\tan t) = t$

$\therefore h = \cos t \sin t - t\sin t$

$= \cos t$

$k = \sin t - t\cos t + t\cos t$

$= \sin t$

The coordinates (h,k) for the centre of curvature are $(\cos t, \sin t)$