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The parametric equations of a curve are as given in equations (1) and (2)

$x = \cos t + t\sin t$ ---- (1)

$y = \sin t - t\cos t$

In terms of t determine:

i) an expression for the radius of curvature (R)

ii) expressions for the coordinates (h, k) of the centre of curvature

Solution

$x = \cos t + t\sin t$

$y = \sin t - t\cos t$

to find $\frac{dx}{dt}$

Let $u = \cos t$ and ~~let~~ $v = t\sin t$

$\therefore \frac{dx}{dt} = \frac{du}{dt} + \frac{dv}{dt}$

But to find $\frac{dv}{dt}$

Using product rule

let $w = t$ and $n = \sin t$

$\therefore dw/dt = 1$ and $dn/dt = \cos t$

$\therefore \frac{dv}{dt} = w\frac{dn}{dt} + n\frac{dw}{dt}$

$\frac{dv}{dt} = t(\cos t) + \sin t(1)$

$\frac{dv}{dt} = \sin t + t\cos t$

$\therefore \frac{dx}{dt} = -\sin t + \sin t + t\cos t = t\cos t$

Now

$\frac{dy}{dt}$

Also let $u = \sin t$ and $v = -t\cos t$

$\therefore \frac{du}{dt} = \cos t$

but $\frac{dv}{dt}$

Let $w = -t$ and $m = \cos t$

$\therefore dw/dt = -1$ $\quad dm/dt = -\sin t$

$\therefore \frac{dv}{dt} = w\frac{dm}{dt} + m\frac{dw}{dt}$

$\frac{dv}{dt} = -t(-\sin t) + \cos t(-1)$

$\frac{dv}{dt} = t\sin t - \cos t$

$\therefore \frac{dy}{dt} = \cos t + t\sin t - \cos t$

$\therefore \frac{dy}{dt} = t\sin t$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$\frac{dy}{dx} = \frac{\cancel{t}\sin t}{\cancel{t}\cos t} = \frac{\sin t}{\cos t}$

Now

$\frac{d^2y}{dx^2}$

Using quotient rule

Let $\frac{\sin t}{\cos t} = \frac{u}{v}$

Recall

$\frac{d^2y}{dx^2} = \frac{v\,du/dx - u\,dv/dx}{v^2}$

Since $u = \sin t$ $\quad \therefore du/dt = \cos t$

$v = \cos t$ $\quad dv/dt = -\sin t$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{\cos t(\cos t) - [\sin t \times -\sin t]}{\cos^2 t}\frac{dt}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{\cos^2 t + \sin^2 t}{\cos^2 t}\frac{dt}{dx}$$

$$\frac{d^2y}{dx^2} = \text{Recall } \cos^2 t + \sin^2 t = 1$$

$$\therefore \frac{d^2y}{dx^2} = \frac{1}{\cos^2 t}\frac{dt}{dx}$$

Recall $\frac{dt}{dx} = \frac{1}{t\cos t}$

$$\therefore \frac{d^2y}{dx^2} = \frac{1}{\cos^2 t} \times \frac{1}{t\cos t} = \frac{1}{t\cos^3 t}$$

Recall

$$R = \frac{1}{R} = \frac{(d^2y/dx^2)}{\left[1 + (dy/dx)^2\right]^{3/2}}$$

$$\therefore R = \frac{\left[1 + (dy/dx)^2\right]^{3/2}}{(d^2y/dx^2)}$$

$$R = \left[1 + \left(\frac{\sin t}{\cos t}\right)^2\right]^{3/2} \div \frac{1}{t\cos^3 t}$$

$$R = \left[\frac{1}{1} + \frac{\sin^2 t}{\cos^2 t}\right]^{3/2} \div$$

$$R = \left[\frac{\cos^2 t + \sin^2 t}{\cos^2 t}\right]^{3/2} \times t\cos^3 t$$

$$R = \left[\frac{1}{\cos^2 t}\right]^{3/2}$$

$$R = \frac{1}{\left(\sqrt{\cos^2 t}\right)^3} \times t\cos^3 t$$

$$R = \frac{1}{\cancel{\cos t^3}} \times t\,\cancel{\cos^3 t}$$

$$R = t$$

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ii) (h, k)

Recall; $h = x_1 - R\sin\theta$ - - - (1)

$k = y_1 + R\cos\theta$ - - - (2)

$R = t$ $\theta = t$

$x_1 = \cos t + t\sin t$

$y_1 = \sin t - t\cos t$

Substituting R, x_1, y_1 into h and k [i.e eqn (1) and (2)]

$h = \cos t + t\sin t - t\sin t$

$h = \cos t$

$k = \sin t - t\cos t + t\cos t$

$k = \sin t$

Therefore the co-ordinates for the centre can be given as
$(\cos t, \sin t)$