DERRI COLUMBUS BOMATE
15/ENG02/017
COMPUTER ENGINEERING

1 ii. If $y = e^{x^2+x}$

Show that

$y'' = y'(2x+1) + 2y$

and, hence, prove that

$y^{(n+2)} = (2x+1)\, y^{(n+1)} + 2(n+1)\, y^{n}$

Solution

$y = e^{x^2+x}$ ---- ①

Taking the first derivative

$dy/dx = (2x+1)\, e^{x^2+x} \Rightarrow y'$ ---- ②

$d^2y/dx^2 = (2x+1)(2x+1)\, e^{x^2+x} + e^{x^2+x}(2)$

$d^2y/dx^2 = (4x^2+4x+1)\, e^{x^2+x} + 2e^{x^2+x}$ ---- ③

Equating ③ to ② and ①

$(4x^2+4x+1)\, e^{x^2+x} = (2x+1)\, e^{x^2+x}(2x+1) + 2(e^{x^2+x})$

$(4x^2+4x+1)\, e^{x^2+x} = (4x^2+4x+1)\, e^{x^2+x} + 2e^{x^2+x}$ ---- ④

$\therefore\ y'' = y'(2x+1) + 2y$

iii. $y'' = y'(2x+1) + 2y$

using Leibnits theorem

$-y'' + y'(2x+1) + 2y = 0$

$y'' \Rightarrow d^2y/dx^2$

$u = y''$, $u^n = y^{(n+2)}$

$v = 1$, $v' = 0$

$y^{(n)} = y^{(n+2)} + 0 \Rightarrow y^{(n+2)}$ ---- ①

$u = y''$, $u^{(n)} = y^{(n+1)}$

$v = 2x+1$, $v' = 2$, $v'' = 0$

$y^{(n)} = y^{(n+1)}\, 2x+1 + ny^{(n)}\, 2 + 0$ ---- ②

$u = y$, $u^n = y^{(n)}$

$v = 2$, $v' = 0$

$y^{(n)} = y^{(n)}\, 2 + 0 \Rightarrow 2y^{(n)}$ ---- ③

Combining the three equations

$y^{(n+2)} = y^{(n+1)}\, 2x+1 + 2ny^{(n)} + 2y^{(n)}$

$y^{(n+2)} = (2x+1)\, y^{(n+1)} + 2y^{(n)}(n+1)$

2\. Using the Leibnitz theorem, given that

i. $y = x^3 e^{4x}$, determine $y^{(5)}$

Solution

$y' = x^3(4e^{4x}) + e^{4x}(3x^2)$

$y' = x^3 4e^{4x} + 3x^2 e^{4x} \quad \ldots\ldots (1)$

$y'' = x^3(16e^{4x}) + 4e^{4x} 3x^2 + 3x^2 4e^{4x} + 6x e^{4x}$

$y''' = x^3 64e^{4x} + 16e^{4x} 3x^2 + 3x^2 16e^{4x} + 6x 4e^{4x} + 3x^2 16e^{4x} + 6x 4e^{4x}$
$+ 6x 4e^{4x} + 6e^{4x}$

$y^{iv} = x^3 256e^{4x} + 3x^2 64e^{4x} + 3x^2 64e^{4x} + 6x 16e^{4x} + 3x^2 64e^{4x} + 3x^2 64e^{4x}$
$+ 6x 16e^{4x} + 96e^{4x} + 3x^2 64e^{4x} + 6x 16e^{4x} + 6x 16e^{4x} + 24e^{4x} + 6x 16e^{4x}$
$+ 24e^{4x} + 24e^{4x} + 0$

$y^{v} = x^3 1024e^{4x} + 3x^2 256e^{4x} + 3x^2 256e^{4x} + 6x 64e^{4x} + 3x^2 256e^{4x} + 6x 64e^{4x}$
$+ 6x 64e^{4x} + 96e^{4x} + 3x^2 256e^{4x} + 6x 64e^{4x} + 96e^{4x} + 384e^{4x} + 0 + 3x^2 256e^{4x}$
$+ 6x 64e^{4x} + 6x 64e^{4x} + 96e^{4x} + 6x 64e^{4x} + 96e^{4x} + 96e^{4x} + 0 + 6x 64e^{4x}$
$+ 96e^{4x} + 96e^{4x} + 0 + 96e^{4x} + 0$

ii. $x^2 \dfrac{d^2y}{dx^2} + x \dfrac{dy}{dx} + y = 0$

show that $x^2 y^{(n+2)} + (2n+1) x y^{(n+1)} + (n^2+1) y^{(n)} = 0$

SOLUTION

$w = x^2 y''$

$v = x^2, \; v' = 2x, \; v^2 = 2, \; v^3 = 0$

$u = y'', \; u^n = y^{(n+2)}$

$y^{(n)} = y^{(n+2)} x^2 + n y^{(n+2-1)} - 2x + \dfrac{n(n-1)}{2} y^{(n+2-2)} 2 + 0$

$= y^{(n+2)} x^2 + 2nx y^{(n-1)} + n(n-1) y^{(n)}$

$w = x y'$

$v = x, \; v' = 1, \; v^2 = 0$

$w^{(n)} = y^{(n+1)} x + n y^{(n+1-1)} + 0$

$= y^{(n+1)} x + n y^n$

$w = y$

$v = 1, \; v' = 0$

$u = y \quad u^{(n)} = y^{(n)}$

$w^{(n)} = y^{n}$

Then $\left[x^2 y'' + xy' + y\right]^n = 0$ becomes

$y^{(n)} = x^2 y^{(n+2)} + 2xn\,y^{(n+1)} + xy^{(n+1)} + n(n-1)\,y^{(n)} + ny^{(n)} + y^{(n)}$

$y^{(n)} = x^2 y^{(n+2)} + (2n+1)\,xy^{(n+1)} + (n^2-n)\,y^{(n)} + ny^{(n)} + y^{(n)}$

$y^{(n)} = x^2 y^{(n+2)} + (2n+1)\,xy^{(n+1)} + (n^2-n+n+1)\,y^{(n)}$

$= x^2 y^{(n+2)} + (2n+1)\,xy^{(n+1)} + (n^2+1)\,y^{(n)} = 0$