15/ENG06/015
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① if $y = e^{x^2+x}$
Show that
$y'' = y'(2x+1) + 2y$

$y = e^{x^2+x}$

$\frac{dy}{dx} = y'$

Let $u = x^2 + x$

$\frac{du}{dx} = 2x + 1$

$\therefore y = e^u$, $\frac{dy}{dx} = y' = dy/du \times \frac{du}{dx}$

$\frac{dy}{dx} = y' = e^u \times 2x + 1$

$y' = (2x+1)e^u$
$y'' = (2x+1)e^{x^2+x}$
$y' = e$

$\frac{d^2y}{dx^2} = y'' =$

$y' = (2x+)e^{x^2+x}$
using product rule
Let $u = 2x + 1$

$\frac{du}{dx} = 2$

$v = e^{x^2+x}$
using chain rule
$\frac{dv}{dx} = 2x + e^{x^2+x}$

$\therefore \frac{d^2y}{dx^2} = y'' = U^{du}/dx + U^{dv}/dx$
$2(e^{x^2+x})\, 2x+1(2x+1\, e^{x^2+x})$
Since $y = e^{x^2+x}$
$y' = (2x+1)e^{x^2+x}$
$y'' = 2(y) + (2x+1)(y')$

$y'' = y'(2x+1) + 2y$

(b) Hence prove that

$$y^{(n+2)} = (2x+1)y^{(n+1)} + 2(n+1)y^{n}$$

Solution

$y'' = -y'(2x+1) + 2y$

$y'' - y'(2x+1) - 2y = 0$

using Leibnitz theorem

① y''

$u = y''$ $\quad$ $v = 1$

$u^{n} = y^{n+2}$ $\quad$ $v' = 0$

$u^{n}v + nu^{n-1}v'$

$y^{n+2}(1) + n(y^{n+1})(0)$

y^{n+2}

② $-y'(2x+1)$ $\quad$ $v = -(2x+1)$

$u = y'$ $\quad$ $v' = -2$

$u^{n} = y^{n+1}$ $\quad$ $v'' = 0$

$u^{n}v + nu^{n-1}v' + n(n-1)u^{n-2}v''$

$= y^{n+1}(-2x+1) + n(y^{n})(-2)$

$= -(2x+1)y^{n+1} - 2ny^{n}$

③ $-2y$

$u = y$ $\quad$ $v = -2$

$u^{n} = y^{n}$ $\quad$ $v = 0$

$= u^{n}v + nu^{n-1}v'$

$= -2y^{n}$

$y^{n+2} - (2x+1)y^{n+1} - 2ny^{n} - 2y^{n} = 0$

$y^{n+2} = (2x+1)y^{(n+1)} + 2y^{n}(n+1)$

$$y^{(n+2)} = (2x+1)y^{(n+1)} + 2(n+1)y^{n}$$

② Using the Leibnitz theorem, given that

(i) $y = x^3 e^{4x}$

determine $y^{(5)}$

$v = x^3$ $\quad$ $u = e^{4x}$

$v' = 3x^2$ $\quad$ $u' = 4e^{4x}$

$v^{(2)} = 6x$ $\quad$ $u'' = 16e^{4x}$

$v^{(3)} = 6$ $\quad$ $u''' = 64e^{4x}$

$v^{(4)} = 0$ $\quad$ $u^{4} = 256e^{4x}$

$v^{(5)} = 0$ $\quad$ $u^{5} = 1024e^{4x}$

$$y^5 = u^5 v + n u^4 v' + \frac{n(n-1)}{2!} u^3 v^2 + \frac{n(n-1)(n-2)}{3!} u^2 v^3 + \frac{n(n-1)(n-2)(n-3)}{4!} u' v^4 + \frac{n(n-1)(n-2)(n-3)(n-4)}{5!} u v^5$$

$$y^5 = [1024e^{4x}(x^3)] + [5(256e^{4x})3x^2] + \left[\frac{5\times4}{2} \times 64e^{4x} \times 6x\right]$$

$$+ \left[\frac{5\times4\times3}{3\times2} \times 16e^{4x} \times 6\right] + [0] + [0]$$

$$y^5 = 1024e^{4x}x^3 + 3840e^{4x}x^2 + 3840e^{4x}x + 960e^{4x}$$

$$y^5 = e^{4x}(1024x^3 + 3840x^2 + 3840x + 960)$$

(ii) $x^3 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = 0$

$x^2 y'' + xy + y = 0$

① $x^2 y''$

$u = y''$ $\quad$ $v = x^2$

$u^n = y^{n+2}$ $\quad$ $v' = 2x$

$v'' = 2$

$v''' = 0$

② $\cancel{2y'}$

$y^n = u^n v + n u^{n-1} v' + \frac{n(n-1)}{2!} u^{n-2} v'' + \frac{n(n-1)(n-2)}{3} u^{n-3}$

v'''

$y^n = y^{n+2} x^2 + n y^{n+1} (2x) + \frac{n(n-1)}{2!} y^n (2) + 0$

$y^n = y^{n+2} x^2 + 2xny^{n+1} + n(n-1) y^n$

③ xy'

$u = y'$ $\quad v = x$

$u^n = y^{n+1}$ $\quad v' = 1$

$v'' = 0$

$y^n = u^n v + n u^{n-1} v + \frac{n(n-1)}{} u^{n-2} v''$

$y^n = y^{n+1} x + n y^n (1) + 0\,2!$

$y^n = x y^{n+1} + n y^n$

④ y

$u = y$ $\quad v = 1$

$u^n = y^n$ $\quad v' = 0$

$y^n = u^n v + \cancel{un^n}\, u n^{n-1} v'$

$= y^n (1) + n (y^{n-1})(0)$

$= y^n$

$y^n = y^{n+2} x^2 + n y^{n+1}\, 2x + (n(n-1) y^n + x y^{n+1} + n y^n + y^n$

$y^n = x^2 (y^{n+2}) + 2xn (y^{n+1}) + (n^2 - n) y^n + x (y^{n+1}) + n y^n + y^n$

$y^n = (n^2 - n + n + 1) y^n + (2xn + x) y^{n+1} + x^2 (y^{n+2})$

$0 = x^2 y(n+2) + (2xn + x) y^{n+1} + (n^2 + 1) y^n$

$0 = x^2 y(n+2) + (2n+1) x y^{(n+1)} + (n^2+1) y^n$

$x^2 y^{(n+2)} + (2n+1) x y^{(n+1)} + (n^2+1) y^n = 0$