1) $y = e^{x^2+x}$

$y' = (2x+1)\, e^{x^2+x}$

$\cancel{y'' = (2x+1)(2x+1)\, e^{x^2+x}}$

$\cancel{y' = (2x+1)\, y^0}$

$\cancel{y'' = (2x+1)\, y'}$

$y'' = 2\, e^{x^2+x} + (2x+1)(2x+1)\, e^{x^2+x}$

$y'' = 2y + (2x+1)\, y'$

Hence

$y'' = y'(2x+1) + 2y$

$y^{(n+2)} = y^{(n+1)} \cdot (2x+1) + (n+1)\, y^n \cdot 2$

$y^{(n+2)} = (2x+1)\, y^{(n+1)} + 2(n+1)\, y^n$

2) i $y = x^3 e^{4x}$

$$y^{(n)} = u^n v + n u^{(n-1)} v' + \frac{n(n-1)}{2!} u^{(n-2)} v^2 + \cdots\cdots$$

Let $x^3 = v$ and $e^{4x} = u$

$\cancel{y_5} =$

$$y^5 = (4)^5 e^{4x} \cdot x^3 + 5(4)^4 e^{4x} \cdot 3x^2 + 10(4)^3 e^{4x} \cdot 6x + \cancel{10(4)^2 e^{4x}}$$
$$+ 10(4)^2 e^{4x} \cdot 6 + 0$$

$$y^5 = 1024x^3 e^{4x} + 3840x^2 e^{4x} + 3840x e^{4x} + 960 e^{4x} + 0$$

$$y^5 = e^{4x}(1024x^3 + 3840x^2 + 3840x + 960)$$

ii) $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

$$x^2 y'' + x y' + y^0 = 0$$
$$x^2 y^{(2)} + x y^{(1)} + y^{(0)} = 0$$

using Leibnitz Theorem

$$y^n = y^{(n+2)} \cdot x^2 + n \cdot 2x y^{(n+1)} + \frac{2n(n-1)}{2!} y^n + y^{(n+1)} \cdot x$$
$$+ n \cdot y^{(n)} \cdot 1 + y^n$$

$$y^n = x^2 y^{(n+2)} + 2x y^{(n+1)} \cdot n + n(n-1) y^n + x y^{(n+1)}$$
$$+ n y^{(n)} + y^{(n)}$$

$$y^n = y^{(n+2)}(x^2) + y^{(n+1)}(2xn + x) + y^{(n)}(n(n-1) + n + 1)$$

$$y^n = x^2 y^{(n+2)} + (2n+1) x y^{(n+1)} + (n^2 - n + n + 1) y^{(n)}$$

$$y^n = x^2 y^{(n+2)} + (2n+1) x y^{(n+1)} + (n^2 - 1) y^{(n)}$$

Hence

$$x^2 y^{(n+2)} + (2n+1) x y^{(n+1)} + (n^2 - 1) y^{(n)} = 0$$