Question 1: $y = e^{x^2+x}$

Show that $y'' = y'(2x+1) + 2y$ and prove that $y^{(n+2)} = (2x+1)y^{(n+1)} + 2(n+1)y^n$

Solution

$y = e^{x^2+x}$ ——— ①

$\therefore y^n = C_1^n e^{ax}$ $y' = (2x+1)e^{x^2+x}$ ——— ②

where $\dfrac{u\,dv}{dx} + \dfrac{v\,du}{dx}$

$u = 2x+1$; $\dfrac{du}{dx} = 2$;

$v = e^{x^2+x}$; $\dfrac{dv}{dx} = (2x+1)e^{x^2+x}$

$y'' = (2x+1)(2x+1)e^{x^2+x} + 2e^{x^2+x}$

From equ ① and ②

$y'' = y'(2x+1) + 2y$

Let $w_1 = y'$ $\qquad w = y'(2x+1)$

$u = y^2$; $v = 1$

$u^n = y^{(2+n)}$

$u = y^{(1)}$ $\qquad v = 2x+1$

$u^n = y^{(1+n)}$ $\qquad v' = 2$

$u^{n-1} = y^n$

$w_2 = 2y$

$u = y$ $\qquad v = 2$

$u^n = y^{(n)}$

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$-\ w^{(n)} = w_2^{(n)} + w_3^{(n)}$

$y^n = u^{(n)}v + n u^{(n-1)} v'$

$y^{(2+n)} = y^{(1+n)} \cdot (2n+1) + n(y^{(n)})' \cdot 2 + y^{(n)} 2$

$y^{(2+n)} = (2n+1) y^{(1+n)} + 2(n+1) y^{(n)}$

Question 2

$y = x^3 e^{4x}$; determine $y^{(5)}$

Solution

$u = e^{4x}$ $\qquad$ $v = x^3$

$u^n = 4^n e^{4x}$ $\qquad$ $v' = 3x^2$

$u^{n-1} = 4^{n-1} e^{4x}$ $\qquad$ $v'' = 6x$

$u^{n-2} = 4^{n-2} e^{4x}$ $\qquad$ $v''' = 6$

$u^{n-3} = 4^{n-3} e^{4x}$

$$y^{(n)} = u^n v + n u^{(n-1)} v' + \frac{n(n-1) u^{(n-2)} v''}{2!} + \frac{n(n-1)(n-2) u^{(n-3)}}{3!} v'''$$

$$y^{(5)} = 4^5 e^{4x} \cdot x^3 + 5(4^4 e^{4x}) \cdot 3x^2 + \frac{5(4)(4^3 e^{4x})}{2!} \cdot 6x^3 + \text{---} +$$

$$\frac{5(4)(3)(4^2 e^{4x}) \cdot \cancel{6}}{\cancel{3!}}$$

$y^{(5)} = 1024x^3 + 5840 x^2 e^{4x} + 3840 x e^{4x} + 96 e^{4x}$

$y^{(5)} = 64 e^{4x} [16x^3 + 60x^2 + 60x + 15]$

ii) $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

show that $x^2 y^{(n+2)} + (2n+1) x y^{(n+1)} + (n^2+1) y^n = 0$

Solution

Let $w = x^2 y$

$u = y^{(2)}$; $u' = y'''$; $u'' = y^{iv}$ $\cdots$ $-\ u^{(n)} = y^{(n+2)}$

$v = x^2$

$v' = 2x$

$v'' = 2$

$v''' = 0$

$$w^n = u^n v^0 + \frac{n u^{n-1}}{1} v' + \frac{n(n-1)}{1 \cdot 2} u^{n-2} v'' + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3} u^{n-3} v''' + \dots$$

$$w^n = y^{n+2} x^2 + n \cdot y^{n+1} \cdot 2x + \frac{n(n-1)}{2} y^n \cdot 2 + \cancel{\frac{n(n-1)(n-2)}{6} y^{n-1}} \cdot 0$$

$$w^n = y^{n+2} \cdot x^2 + n \cdot 2x y^{n+1} + n(n-1) y^n$$

Let $w = xy'$

$u = y'$; $u' = y''$; $u^n = y^{(n+1)}$

$v = x$

$v' = 1$

$v'' = 0$

$$w^{(n)} = u^n v^0 + \frac{n u^{n-1} v'}{1} + \frac{n(n-1) u^{n-2} v''}{1 \cdot 2}$$

$$w^{(n)} = y^{n+1} + x + \frac{n y^n}{1} \cdot 1 + \cancel{\frac{n(n-1) y^{n-1}}{1\cdot 2}} \cdot 0$$

$$w^{(n)} = y^{n+1} + x + n y^n \cdot 1$$

where $w = y$

$w^n = y^n$

$\therefore y^{n+2} x^2 + n y^{n+1} \cdot 2x + n(n-1) y^n + y^{n+1} + x + n y^n + y^n = 0$

$= y^{n+2} x^2 + [n \cdot 2x y^{n+1} + x y^{n+1}] + [n(n-1) y^n + n y^n + y^n] = 0$

$= x^2 y^{n+2} + (2n+1) x y^{n+1} + (n(n-1) + 1 + n) y^n = 0$

$= x^2 y^{n+2} + (2n+1) x y^{n+1} + (n^2 - n + 1 + n) y^n = 0$

$= x^2 y^{n+2} + (2n+1) x y^{n+1} + (n^2+1) y^n = 0$