

1) If $y = e^{x^2+x}$, show that $y' = y(2x+1) + 2y$ and hence, prove that $y^{(n+2)} = (2x+1)y^{(n+1)} + 2(n+1)y^n$

Soln

$$y = e^{x^2+x}$$

$$\frac{dy}{dx} = (2x+1)e^{x^2+x}$$

$$\begin{aligned} \frac{d^2y}{dx^2} &= 2e^{x^2+x} + (2x+1)(2x+1)e^{x^2+x} \\ &= e^{x^2+x} [(2x+1)(2x+1) + 2] \cdot y'(2x+1) + 2y \\ &= (2x+1)e^{x^2+x} (2x+1) + 2y \cdot (2x+1)e^{x^2+x} \\ &\quad (2x+1) + 2e^{x^2+x} \end{aligned}$$

$$\frac{d^2y}{dx^2} = e^{x^2+x} [(2x+1)(2x+1) + 2]$$

$$y'' = y'(2x+1) + 2y$$

Let d_1, d_2 & d_3 represent y'', y' & y

$$d_1, \quad v = y^2 \\ v^n = y^{(n+2)}$$

$$d_2, \quad u = y' \quad v = 2x+1 \\ u^n = y^{(n+1)} \quad v' = 2 \\ u^{n-1} = y^n \quad v'' = 0$$

$$d_3, \quad u = y \quad v = 2 \\ u^n = y^n \quad v' = 0$$

Using $d_1 = d_2 + d_3$

$$y^{n+2} = y^{n+1}(2x+1) + ny^n \cdot 2 + y^{n+2}$$

$$y^{n+2} = y^{n+1}(2x+1) + 2ny^n + 2y^n$$

$$y^{n+2} = (2x+1)y^{(n+1)} + 2(n+1)y^n$$

2) using the Leibnitz theorem, given that

a) $y = x^3 e^{4x}$, determine $y^{(n)}$

b) $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = 0$, show that

show that,

$$x^2 y^{(n+2)} + (2n+1)x y^{(n+1)} + (n^2+1)y^{(n)} = 0.$$

Soln

a) $y = x^3 e^{4x}$

$V = x^3$

$u = e^{4x}$

$V' = 3x^2$

$u^n = 4^n e^{4x}$

$V'' = 6x$

$u^{n-1} = 4^{n-1} e^{4x}$

$V''' = 6$

$u^{n-2} = 4^{n-2} e^{4x}$

$V^{IV} = 0$

$u^{n-3} = 4^{n-3} e^{4x}$

$y^n = 4^n e^{4x} x^3 + n 4^{n-1} e^{4x} \cdot 3x^2 + \frac{n(n-1)}{2!} 4^{n-1} e^{4x} \cdot 6x$

$+ \frac{n(n-1)(n-2)}{3!} 4^{n-2} e^{4x} \cdot 6$

When $n=5$

$y^{(5)} = 4^5 e^{4x} x^3 + (5)(4^4) e^{4x} \cdot 3x^2 + \frac{5(5-1)}{2!} 4^{5-1} e^{4x} \cdot 6x$

$+ \frac{5(5-1)(5-2)}{3!} 4^{5-2} e^{4x} \cdot 6$

$y^{(5)} = 4^5 e^{4x} x^3 + 20^4 e^{4x} \cdot 3x^2 + (10) 256 e^{4x} \cdot 6x + (10)(16) e^{4x} \cdot 6$

$y^{(5)} = 4^5 e^{4x} x^3 + 20^4 e^{4x} \cdot 3x^2 + 2560 e^{4x} \cdot 6x + 160 e^{4x} \cdot 6$

b) $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = 0$

$x^2 (y'') + xy' + y = 0$

Let d_1, d_2 & d_3 represent y'' , y' & y

$d_1,$

$u = y(x^2)$

$V = x^2$

$u^n = y^{n+2}$

$V' = 2x$

$u^{n-1} = y^{n+1}$

$V'' = 2$

$u^{n-2} = y^n$

$V''' = 0$

$d_2,$

$u = y(x)$

$V = x$

$u^n = y^{n+1}$

$V' = 1$

$u^{n-1} = y^n$

$V'' = 0$

d3,

$$u = y$$
$$u^n = y^n$$

using

$$d_1 + d_2 + d_3 = 0$$

$$y^{n+2} \cdot x^2 + n y^{n+1} \cdot 2x + \frac{n(n-1)}{2!} y^{n-2} + y^{n+1} \cdot x + n y^n$$

$$+ y^n = 0$$

$$x^2 y^{(n+2)} + 2x n y^{(n+1)} + n(n-1) y^n + x y^{n+1} + n y^n + y^n$$

$$= 0$$

$$x^2 y^{(n+1)} + 2x n y^{(n+1)} + x y^{(n+1)} + n(n-1) y^n + n y^n + y^n$$

$$= 0$$

$$x^2 y^{(n+1)} + x y^{(n+1)} (2n+1) + y^n [n(n-1) + (n+1)] = 0$$

$$\cancel{x^2 y^{(n+1)}} + x y^{(n+1)} (2n+1) + y^n$$

$$x^2 y^{(n+1)} + x y^{(n+1)} (2n+1) + y^n (n^2+1) = 0$$