① $y = e^{x^2+x}$

$y'' = y'(2x+1) + 2y$

$\frac{dy}{dx} = (2x+1)e^{x^2+x}$

$y'' = (2x+1)(2x+1)e^{x^2+x} + 2e^{x^2+x}$

$= e^{x^2+x}((2x+1)^2+2)$

$\therefore e^{x^2+x}((2x+1)^2+2) = e^{x^2+x}((2x+1)(2x+1)+2)$

$y'' = y'(2x+1) + 2y$

$y'' = y'(2x+1) + 2y$

$y'' - y'(2x+1) - 2y = 0$

First sub: $W = y''$

$v = 1, v' = 0$

$u = y^2, y^{(n)} = y^{(n+2)}$

~~Second sub~~ $y^{(n)} = y^{(n+2)}$

Second sub: $W = -y'(2x+1)$

$v = 2x+1, v' = 2, v'' = 0$

$u = -y', u^n = -y^{(n+1)}$

$y^{(n)} = -y^{(n+1)}(2x+1) + n(-y^{(n+1)}) \cdot 2$

$= -y^{(n+1)}(2x+1) + 2n(-y^{(n)})$

~~3rd~~ 3rd sub: $W = -2y$

$v = -2, v' = 0$

$u = y, u^{(n)} = y^{(n)}$

$y^{(n)} = -2y^{(n)} + 0$

$y^{(n)} - y^{(n+2)} - y^{(n+1)}(2x+1) + 2n(-y^{(n)}) - 2y^n$

$$y^{(n+2)} = (2x+1)y^{(n+1)} + (2n+2)y^{(n)}$$

$$y^{(n+2)} = (2x+1)y^{(n+1)} + 2(n+1)y^{(n)}$$

Assignment 3

(10) $y = x^3 e^{4x}$ $y^5 = ?$

$V = x^3$, $V' = 3x^2$, $V'' = 6x$, $V''' = 6$, $V'^V = 0$

$U = e^{4x}$, $U^{(n)} = 4e^{4x}$, $U'' = 16e^{4x}$, $U''' = 64e^{4x}$, $U'^V = 256e^{4x}$, $U^V = 1024e^{4x}$

$$y^5 = U^{(n)}V + nU^{(n-1)}V' + \frac{n(n-1)}{2!}U^{(n-2)}V'' + \frac{n(n-1)(n-2)}{3!}U^{(n-3)}V''' \ldots$$

$$y^{(5)} = 1024e^{4x}[x^3] + 15x^2[256e^{4x}] + 60x(64e^{4x}) + 60(16e^{4x})$$

$$y^{(5)} = 64e^{4x}(16x^3 + 60x^3 + 60x + 15)$$

(11) $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

$x^2y'' + xy' + y = 0$

first sub:- $V = x^2$, $V' = 2x$, $V'' = 2$, $V''' = 0$

$U = y''$, $U^{(n)} = y^{(n+2)}$

$$y^{(n)} = y^{(n+2)}x^2 + ny^{(n+1)}2x + \frac{n(n-1)}{2!}y^{(n)}2$$

Second sub:- $V = x$, $V' = 1$, $V'' = 0$

$U = y'$, $U^{(n)} = y^{(n+1)}$

$y^{(n)} = y^{(n+1)}x + ny^{(n)}$

Third sub:- $V = 1$, $V' = 0$

$U = y$, $U^{(n)} = y^{(n)}$

$y^{(n)} = y^{(n)} \cdot 1$

$x^2y'' + xy' + y = 0$

$y^{(n+2)}x^2 + ny^{(n+1)}2x + \frac{n(n-1)}{2!}y^{(n)}2 + y^{(n+1)}x + ny^{(n)} + y^{(n)} = 0$

$x^2y^{(n+2)} + 2xny^{(n+1)} + n(n-1)y^{(n)} + xy^{(n+1)} + ny^{(n)} + y^{(n)} = 0$

$x^2y^{(n+2)} + 2xny^{(n+1)} + xy^{(n+1)} + (n^2-n)y^{(n)} + ny^{(n)} + y^{(n)} = 0$

$$x^2 y^{(n+2)} + (2xn + x) y^{(n+1)} + (n^2 - n + n) y^{(n)} = 0$$

$$x^2 y^{(n+2)} + (2n+1) x y^{(n+1)} + (n^2+1) y^{(n)} = 0$$