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Chemical Engineering
Eng 381

1. If $y = e^{x^2+x}$, show that $y'' = y'(2x+1) + 2y$ and hence prove that $y^{(n+2)} = (2x+1)y^{(n+1)} + 2(n+1)y^n$

Solution

$y = e^{x^2+x}$
$u = e^{x^2+x}$ $\quad v = 1$
$u^n = (2x+1)^n e^{x^2+x}$ $\quad u^{(1)} = 0$
$u^{(n-1)} = (2x+1)^{n-1} e^{x^2+x}$
$y^n = u^n v + nu^{(n-1)} v'$
$y^n = (2x+1)^n e^{x^2+x} . 1 + (2x+1)^{n-1} e^{x^2+x} . 0$
$y^n = (2x+1) e^{x^2+x}$, let $n = 1$
$y^{(1)} = (2x+1)' e^{x^2+x}$
$u = e^{x^2+x}$ $\quad v = 2x+1$
$u^{(n)} = (2x+1)^n e^{x^2+x}$ $\quad v' = 2$
$u^{n-1} = (2x+1)^{n-1} e^{x^2+x}$ $\quad v^2 = 0$
$y^{(n)} = (2x+1)^n e^{x^2+x} (2x+1) + n(2x+1)^{n-1} e^{x^2+x} . 2$
$= (2x+1)(2x+1)^n e^{x^2+x} + 2n(2x+1)^{n-1} e^{x^2+x}$.
Let $n = 1$, $y'' = (2x+1)(2x+1)^n e^{x^2+x} + 2n(2x+1)^{n-1} e^{x^2+x}$
$y'' = (2x+1)(2x+1)' e^{x^2+x} + 2(e^{x^2+x})$
Sub. $y = e^{x^2+x}$ and $y' = (2x+1) e^{x^2+x}$ in y''
$y'' = (2x+1)y' + 2y$
$y'' = y'(2x+1) + 2y$.

ii $y^{(2)} = y_{(1)}(2x+1) + 2y$
$y^{(2)} - y^{(1)}(2x+1) - 2y = 0$
$w_1 = y^{(2)}$
$w_2 = -y^{(1)}(2x+1)$
$w_3 = -2y$
$u = y^2$ $\quad v = 1$; $u = -y'$ $\quad v = 2x+1$; $-u = y'$ $\quad v = 2$
$u^n = y^{(n+2)}$ $\quad v^{(1)} = 0$, $u^n = -y^{(n+1)}$ $\quad v^{(1)} = 2$; $u^n = y^n$ $\quad v' = 0$
$u^{n-1} = y^n$ $\quad v'' = 0$
$y^{(n)=0} = u^{(n)} + nu^{(n-1)} v^{(1)}$
$y^{(n)=0} = y^{(n+2)}(1) - y^{(n+1)}(2x+1) + n(-y^n)(2x) + 2(-y^n)$

$y^{(n)=0}$

$y^{(n)=0} = y^{(n+2)} - (2x+1)y^{(n+1)} - 2xn(-y^{n}) - 2(y^{n})$

$+ 2n(y^{n}) + 2(y^{n}) + (2x+1)y^{(n+1)} = y^{(n+2)}$

$y^{(n+2)} = (2x+1)y^{n-1} + 2(n+1)y^{(n)}$

2 Using Leibnitz theorem

$y = x^3 e^{4x}$, determine $y^{(5)}$

Solution

$x = e^{4x} \qquad V = x^3$

$y^{n} = U^{n}V + nU^{(n-1)}V^{1} + \frac{n(n-1)}{2!}U^{(n-2)}V^{2} + \frac{n(n-1)(n-2)}{3!}U^{(n-3)}V^{3} + \cdots$

$y^{5} = U^{5}V + 5U^{(5-1)}V^{1} + \frac{5(5-1)}{2!}U^{(5-2)}V^{2} + \frac{5(5-1)(5-2)}{3!}U^{(5-3)}V^{3} +$

$\frac{5(5-1)(5-2)(5-3)}{4!}U^{(5-4)}V^{4}$

$U = e^{4x} \qquad V = x^3$

$U^{1} = 4e^{4x} \qquad V^{1} = 3x^2$

$U^{2} = 16e^{4x} \qquad V^{2} = 6x$

$U^{3} = 64e^{4x} \qquad V^{3} = 6$

$U^{4} = 256e^{4x} \qquad V^{4} = 0$

$U^{5} = 1024e^{4x} \qquad V^{5} = 0$

$y^{5} = 1024e^{4x} \cdot x^3 + 5U^{4}V^{1} + \frac{5(4)}{2}U^{3}V^{2} + \frac{5(4)(3)}{6}U^{2}V^{3} +$

$\frac{5(4)(3)(2)}{24}U^{1}V^{4}$

$y^{5} = 1024e^{4x} \cdot x^3 + 5(256e^{4x})3x^2 + \frac{5(4)}{2} \cdot 64e^{4x} \cdot 6x + \frac{5(4)(3)}{6}$

$\cdot 16e^{4x} \cdot 6 + \frac{5(4)(3)(2)}{24} 4e^{4x} \cdot 0$

$y^{5} = 1024e^{4x}x^3 + 3840e^{4x}x^2 + 3840e^{4x} + 960e^{4x}$

$y^{5} = 64e^{4x}(16x^3 + 60x^2 + 60x + 15)$

ii $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

$x^2 y'' + xy' + y = 0$

$$x^2 y^{(2)} + x y^{(1)} + y^{(0)} = 0$$

$w_1 = x^2 y^{(2)}$

$w_2 = x y^{(1)}$

$w_3 = y^{(0)}$

for w_1 $\quad U = y^{(2)}$ $\quad V = x^2$

$U^{n} = y^{n+2}$ $\quad V' = 2x$

$U^{n-1} = y^{n+1}$ $\quad V'' = 2$

$U^{n-2} = y^{n}$ $\quad V''' = 0$

for w_2 $\quad U = y^{(1)}$ $\quad V = x$

$U^{n} = y^{n+1}$ $\quad V' = 1$

$U^{n-1} = y^{n}$ $\quad V'' = 0$

for w_3 $\quad U = y^{(0)}$ $\quad V = 1$

$U^{n} = y^{n}$ $\quad V' = 0$

$$y^{n} = y^{(n+2)} \cdot x^2 + n y^{(n+1)} \cdot 2x + \frac{n(n-1)}{2\times 1} y^{n} x^2 + y^{(n+1)} \cdot x + n y^{n} \cdot 1 + y^{n}$$

$$y^{n} = y^{(n+2)} \cdot x^2 + 2xny^{(n+1)} + n(n-1)y^{n} + xy^{(n+1)} + ny^{n} + y^{n}$$

$$y^{n} = x^2 y^{(n+2)} + 2nxy^{(n+1)} (n^2 - n)y^{(n)} + xy^{(n+1)} + y^{n} + ny^{n}$$

$$y^{n} = x^2 y^{(n+2)} + xy^{(n+1)}(2n+1) + y^{(n)}(n^2 - n + 1 + n)$$

$$y^{n} = x^2 y^{(n+2)} + xy^{(n+1)}(2n+1) + y^{(n)}(n^2+1) = 0$$

$$x^2 y^{(n+2)} + xy^{(n+1)}(2n+1) + y^{(n)}(n^2+1) = 0$$