1) If $y = e^{x^2+x}$ show that $y'' = y'(2x+1) + 2y$ and hence prove that $y^{(n+2)} = (2x+1)y^{(n+1)} + 2^{(n+1)}y$

Soln

$y = e^{x^2+x}$

Let $u = x^2 + x$

$\frac{du}{dx} = 2x+1$

$y = e^u$

$\frac{dy}{du} = e^u$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= e^u \times 2x+1$

where $u = x^2 + x$

$\frac{dy}{dx} = e^{x^2+x} \cdot (2x+1) = y'$

if $y' = e^{x^2+x}(2x+1) = y'$

then $y'' = 2e^{x^2} e^{x^2+x}(2) + [e^{x^2+x} \cdot (2x+1)](2x+1)$

$y'' = 2e^{x^2+x} + e^{x^2+x} \cdot (2x+1) \cdot (2x+1)$

but $y = e^{x^2+x}$

and $y' = e^{x^2+x}(2x+1)$

$y'' = 2(y) + y'(2x+1)$

Applying Leibnitz theorem to the above equation find the nth derivative

$y^{(n+2)} = 2(n+1)y^n + y^{(n+1)}(2x+1)$

2) Using Leibnitz theorem given that

$y = x^3 e^{4x}$ determine $y^{(5)}$

Soln

Recall that Leibnitz theorem states that

$y^5 = u^5 v + 5u^4 v' + 10u^3 v^2 + 10u^2 v^3 + 5u' v^4 + uv^5$

where $u = e^{4x}$ $\quad v = x^3$

$u' = 4e^{4x}$ $\quad v' = 3x^2$

$u^2 = 16e^{4x}$ $\quad v^2 = 6x$

$u^3 = 64e^{4x}$ $\quad v^3 = 6$

$u^4 = 256e^{4x}$ $\quad v^4 = 0$

$u^5 = 1024e^{4x}$

Therefore

$y^5 = 1024e^{4x}(x^3) + 5(256e^{4x})(3x^2) + 10(64e^{4x})(6x) + 10(16e^{4x})6$
$+ 5(4e^{4x})(0)$

$= 1024x^3 \cdot e^{4x} + 3840x^2e^{4x} + 3840xe^{4x} + 960e^{4x}$

$y^5 = e^{4x}(1024x^3 + 3840x^2 + 3840x + 960)$

ii) If $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

show that $x^2y^{(n+2)} + (2n+1)xy^{(n+1)} + (n^2+1)y^n = 0$

soln

The equation can be written as $x^2y'' + xy' + y = 0$

let $w_1 = x^2y''$, $w_2 = xy'$ and $w_3 = y$

solving for $w_1 = x^2y''$

let $u = y^2$, $u^n = y^{n+2}$

let $v = x^2$, $v' = 2x$, $v'' = 2$, $v''' = 0$

$w_1 = y^{(n+2)}x^2 + ny^{(n+1)}2x + \frac{n(n-1)}{2!}y^n \times 2 + 0$

$w_2 = xy'$

let $u = y'$, $u^n = y^n$

let $v = x$, $v' = 1$ and $v'' = 0$

$w_2 = y^{(n+1)} \cdot x + ny^n \times 1 + 0$

$= xy^{n+1} + ny^n$

$w_3 = y^{(n)}$

combining

$w = w_1 + w_2 + w_3$

$w = x^2y^{(n+2)} + ny^{(n+1)} \cdot 2x + \frac{n(n-1)}{2}y^{n \times 2} \cdot xy' + y^{(n+1)}x$

$+ ny^n + y^n = 0$

$= x^2y^{(n+2)} + xy^{(n+1)}(2n+1) + y^n[(n(n-1) + n + 1)] = 0$

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$x^2 y^{(n+2)} + (2n+1) x y^{n+1} + (n^2+1) y^n = 0$