$0 = (n^2 - n + n + 1) y^n + (2xn + x) y^{n+1} + x^2 (y^{n+2})$

$0 = x^2 y^{n+2} + (2xn + x) y^{n+1} + (n^2 + 1) y^n$

$0 = x y^{n+2} + (2n+1) x y^{n+1} + (n^2 + 1) y^n$

$x y^{n+2} + (2n+1) x y^{n+1} + (n^2+1) y^n = 0$

$= u^n v^0 + n u^{n-1} v^1$

$\therefore \quad = -y^{n+1}(2x+1) - 2n\, y^n$

$w^n = -y^{n+1}(2x+1) - 2n y^n$

Let $w = -2y$

$u = y \qquad v = -2$

$u^n = y^n \qquad v' = 0$

$w^n = u^n v + n u^{n-1} v'$

$w^n = -2y^n$

$\therefore \quad y^{n+2} - (2x+1)y^{n+1} - 2ny^n - 2y^n = 0$

$y^{n+2} = (2x+1)y^{n+1} + 2ny^n + 2y^n$

$y^{n+2} = (2x+1)y^{(n+1)} + (2n+2)y^n$

$y^{n+2} = (2x+1)y^{(n+1)} + 2(n+1)y^n$

$y^{n+2} = (2x+1)y^{n+1} + 2y^{(n+1)} + (2n+2)y^n$

2 Using the leibntz theorem given that

$y = x^5 e^{4x}$

$y' = 2(2x+1)\,e^{x^2+x}$

$\frac{d^2y}{dx^2} = y''$

Since $y' = (2x+1)\,e^{x^2+x}$

$\frac{d^2y}{dx^2} = y'' = v\frac{du}{dx} + u\frac{dv}{dx}$

$= 2(e^{x^2+x}) + 2x+1\,(2x+1\,e^{x^2+x})$

since $y = e^{x^2+x}$

$y' = (2x+1)\,e^{x^2+x}$

$y'' = 2(y) + (2x+1)(y')$

$y'' = y'(2x+1) + 2y$.

b) Hence prove that

$y^{(n+2)} = (2x+1)\,y^{n+1} + 2(n+1)\,y^{n}$

$y'' = y'(2x+1) + 2y$

$y'' - y'(2x+1) - 2y = 0$

using leibnitz theorem

Let $w = y''$

$u = y''$

$u^{n} = y^{n+2}$

$w^{n} = y^{n+2}$

Let $w = y'(2x+1)$

① $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

$x^2 y'' + xy' + y = 0$

Let $w = x^2 y''$

$u = y''$ $\quad\quad$ $v = x^2$

$u^n = y^{n+2}$ $\quad\quad$ $v' = 2x$

$v'' = 2$

$v''' = 0$

$$w^n = u^n v^0 + n u^{n-1} v' + \frac{n(n-1)}{2!} u^{n-2} v'' + \frac{n(n-1)(n-2)}{3!} u^{n-3} v'''$$

$$= y^{n+2} x^2 + n y^{n+1}(2x) + \frac{n(n-1)}{2} y^n (2) + 0$$

$$w^n = y^{n+2} x^2 + 2xny^{n+1} + n(n-1) y^n$$

$= (2x+1) y^{(n+1)} + (2n+2) y^n$

$+ 2(n+1) y^n$

$$y^{n+2} - (2x+1)\, y^{n+1} - 2n y^{n} - 2y^{n} = 0$$
$$y^{n+2} = (2x+1)\, y^{n+1} + 2n y^{n} + 2y^{n}$$
$$y^{n+2} = (2x+1)\, y^{(n+1)} + (2n+2)\, y^{n}$$
$$y^{n+2} = (2x+1)\, y^{(n+1)} + 2(n+1)\, y^{n}$$

the Leibnitz theorem given that