Assignment 3

i. 4 $y = e^{x^2+x}$

Show that $y'' = y'(2x+1)+2y$

and hence, prove that $y^{(n+2)} = (2x+1)\, y^{(n+1)} + 2(n+1)y^{n}$

Solution

$y = e^{x^2+x}$ ....(1)

Taking the first derivative

$dy/dx = (2x+1)\, e^{x^2+x} \Rightarrow y'$ ....(ii)

$d^2y/dx^2 = (2x+1)(2x+1)\, e^{x^2+x} + e^{x^2+x}$ ....(ii)

$d^2y/dx^2 = (4x^2+4x+1)\, e^{x^2+x} + 2e^{x^2+x}$ ....(iii)

equating (iii) to (ii) $\cancel{\&}$ (1)

$(4x^2+4x+1)e^{x^2+x} = (2x+1)e^{x^2+x}(2x+1) + 2(e^{x^2+x})$

$(4x^2+4x+1)e^{x^2+x} = (4x^2+4x+1)e^{x^2+x} + 2e^{x^2+x}$ ....(iv)

$y'' = y'(2x+1)+2y$

ii. $y'' = y'(2x+1)+2y$

using Leibnitz theorem

$y'' + y'(2x+1)+2y = 0$

$y'' \Rightarrow d^2y/dx^2$

$u = y''$, $u^n = y^{(n+2)}$

$v = 1$, $v' = 0$

$y^{(n)} = y^{(n+2)} + 0 \Rightarrow y^{(n+2)}$ .. (1)

$u = y''$, $u^{(n)} = y^{(n+1)}$

$v = 2x+1$, $v' = 2$, $v'' = 0$

$y^{(n)} = y^{(n+1)}\, 2x+1 + n y^{(n)}\, 2 + 0$ ....(ii)

$u = y$, $u^n = y^{(n)}$

$v = 2$, $v' = 0$

$y^{(n)} = y^{n}\, 2 + 0 \Rightarrow 2y^{(n)}$ ... (iii)

Combining

$y^{(n+2)} = y^{(n+1)}\, 2x+1 + 2ny^{(n)} + 2y^{(n)}$

$y^{(n+2)} = (2x+1)\, y^{(n+1)} + 2y^{(n)}(n+1)$

2 Using the Leibnitz theorem, given that

i $y = x^3 e^{4x}$, determine $y^{(5)}$

Solution

$y^1 = x^3 (4e^{4x}) + e^{4x}(3x^2)$

$y^1 = x^3 4e^{4x} + 3x^2 e^{4x} \quad \cdots (1)$

$y'' = x^3(16e^{4x}) + 4e^{4x} 3x^2 + 3x^2 4e^{4x} + 6x e^{4x}$

$y''' = x^3 64 e^{4x} + 16e^{4x} 3x^2 + 3x^2 16e^{4x} + 6x 4e^{4x} + 3x^2 16e^{4x}$
$+ 6x 4e^{4x} + 6x 4e^{4x} + 6e^{4x}$

$y^{iv} = x^3 256e^{4x} + 3x^2 64e^{4x} + 3x^2 64e^{4x} + 6x 16e^{4x} + 3x^2$
$64e^{4x} + 3x^2 64e^{4x} + 6x 16e^{4x} + 96e^{4x} + 3x^2 64e^{4x}$
$+ 6x 16e^{4x} + 6x 16e^{4x} + 24e^{4x} + 6x 16e^{4x} + 24e^{4x}$
$+ 24e^{4x} + 0$

$y^v = x^3 1024e^{4x} + 3x^2 256e^{4x} + 3x^2 386e^{4x} + 6x 64e^{4x}$
$+ 3x^2 256e^{4x} + 6x 64e^{4x} + 6x 64e^{4x} + 96e^{4x} +$
$3x^2 256e^{4x} + 6x 64e^{4x} + 96e^{4x} + 384e^{4x} + 0 +$
$3x^3 256e^{4x} + 6x 64e^{4x} + 96e^{4x} + 96e^{4x} + 0$
$+ 96e^{4x} + 0$

ii $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

show that $x^2 y^{(n+2)} + (2n+1)xy^{(n+1)} + (n^2+1)y^{(n)} = 0$

Solution

$w = x^2 y''$

$v = x^2 \quad , v' = 2x \quad , v'' = 2 \quad , v''' = 0$

$u = y'' \quad , u^n = y^{(n+2)}$

$y^{(n)} = y^{(n+2)} x^2 + ny^{(n+2+1)} \cdot 2x + \frac{n(n-1)}{2} y^{(n+1+2)} 2 + 0$

$= y^{(n+2)} x^2 + 2nxy^{(n-1)} + n(n-1)y^{(n)}$

$w = xy'$

$v = x \quad , v' = 1 \quad , v'' = 0$

$w^n = y^{(n+1)} x + ny^{(n+1-1)} + 0$

$= y^{(n+1)} x + ny^n$

$w = y$

$v = 1 \;,\; v' = 0$

$u = y \;,\; u^{(n)} = y^{(n)}$

$w^{(n)} = y^{n}$

Then $(x^2 y'' + xy' + y)^n = 0$ becomes

$y^{(n)} = x^2 y^{(n+2)} + 2xn\,y^{(n+1)} + xy^{(n+1)} + n(n-1)y^{(n)} + ny^{(n)} + y^{(n)}$

$y^{(n)} = x^2 y^{(n+2)} + (2n+1)\,xy^{(n+1)} + (n^2-n)\,y^{(n)} + ny^{(n)} + y^{(n)}$

$y^{(n)} = x^2 y^{(n+2)} + (2n+1)\,xy^{(n+1)} + (n^2-n+n+1)\,y^{(n)}$

$= x^2 y^{(n+2)} + (2n+1)\,xy^{(n+1)} + (n^2+1)\,y^{(n)} = 0$.