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16/ENG07/033

Petroleum Engineering

ENG 381

Assignment 3

① If $y = e^{x^2+x}$, show that $y'' = y'(2x+1) + 2y$ and hence, prove that $y^{(n+2)} = (2x+1)y^{(n+1)} + 2(n+1)y^n$

Solution

$y = e^{x^2+x}$; $u = e^{x^2+x}$; $v = 1$

$u^n = (2x+1)^n e^{x^2+x}$ $\quad v' = 0$

$u^{n-1} = (2x+1)^{n-1} e^{x^2+x}$

$y^{(n)} = u^n v^0 + n u^{(n-1)} v^{(1)}$; $y^n = (2x+1)^n e^{x^2+x}$

let $n = 1$

$y' = (2x+1)e^{x^2+x}$; $u = e^{x^2+x}$; $v = 2x+1$

$u^n = (2x+1)^n e^{x^2+x}$ $\quad V' = 2$

$u^{n-1} = (2x+1)^{n-1} e^{x^2+x}$ $\quad v'' = 0$

$y'^{(n)} = (2x+1)^n e^{x^2+x} \cdot (2x+1) + n(2x+1)^{n-1} e^{x^2+x} \cdot 2$

$= (2x+1)(2x+1)^n e^{x^2+x} + 2n(2x+1)^{n-1} e^{x^2+x}$

let $n = 1$

$y'' = (2x+1)(2x+1)' e^{x^2+x} + 2(2x+1)^0 e^{x^2+x}$

$y'' = (2x+1)(2x+1)' e^{x^2+x} + 2(e^{x^2+x})$

Substitute $y = e^{x^2+x}$ and $y' = (2x+1)' e^{x^2+x}$ in y''

$y'' = (2x+1)y' + 2y$

$y'' = y'(2x+1) + 2y$

This can be written as

$y^{(2)} = y^{(1)}(2x+1)2y$

$y^{(n)} = u^{(n)}v + nu^{(n-1)}v^{(1)}$

$y^{(n+2)} = y^{(n+1)}(2x+1) + \cdots ny^{(n)}\cdot 2 + y^{n}\cdot 2$

$y^{n+2} = (2x+1)y^{n+1} + 2ny^{n} + 2y^{n}$

$y^{n+2} = (2x+1)y^{(n+1)} + 2ny^{n} + 2y^{n}$

$y^{n+2} = (2x+1)y^{n+1} + 2(n+1)y^{n}$

$\boxed{u=y,\ v=2x+1,\ u=y,\ v=2,\ u^{n}=y^{n+1},\ v'=2,\ u^{n}=y^{n},\ v'=0,\ u^{n-1}=y^{n},\ v''=0}$

(2) using the leibnitz theorem given that

(i) $y = x^3 e^{4x}$; determine y^5

Solution

$u = e^{4x}$ $\quad v = x^3$

$u^{n} = 4^{n}e^{4x}$ $\quad v' = 3x^2$

$u^{n-1} = 4^{(n-1)}e^{4x}$ $\quad v'' = 6x$

$u^{n-2} = 4^{(n-2)}e^{4x}$ $\quad v''' = 6$

$u^{n-3} = 4^{(n-3)}e^{4x}$ $\quad v^{IV} = 0$

$$y^{n} = u^{n}v + nu^{(n-1)}v' + \frac{n(n-1)}{2!}u^{(n-2)} + \frac{n(n-1)(n-2)}{3!}u^{(n-3)}v'''$$

$$y^{n} = 4^{n}e^{4x}\cdot x^3 + n4^{(n-1)}e^{4x}\cdot 3x^2 + \frac{n(n-1)\cdot 4^{(n-2)}}{2}e^{4x}\cdot 6x + \frac{n(n-1)(n-2)}{3\times 2}\cdot 4^{(n-3)}e^{4x}\cdot 6$$

$$y^{n} = 4^{n}e^{4x}\cdot x^3 + n4^{(n-1)}e^{4x}\cdot 3x^2 + n(n-1)\cdot 4^{(n-2)}e^{4x}\cdot 3x + n(n-1)(n-2)\cdot 4^{(n-3)}e^{4x}$$

$$y^5 = 4^5 e^{4x}\cdot x^3 + 5\cdot 4^{(5-1)}e^{4x}\cdot 3x^2 + 5(5-1)\cdot 4^{(5-2)}e^{4x}\cdot 3x + 5(5-1)(5-2)\cdot 4^{(5-3)}e^{4x}$$

$y^5 = 1024x^3e^{4x} + 3840x^2e^{4x} + 3840xe^{4x} + 960e^{4x}$

$y^5 = 64e^{4x}(16x^3 + 60x^2 + 60x + 15)$

(ii) $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$, show that $x^2y^{(n+2)} + (2n+1)xy^{(n+1)} + (n^2+1)y^n = 0$

Solution

$x^2y'' + xy' + y = 0$

$x^2y^{(2)} + xy^{(1)} + y^{(0)} = 0$

$u = y''$ $\quad v = x^2$

$u^n = y^{(n+2)}$ $\quad v' = 2x$

$u^{(n-1)} = y^{(n+1)}$ $\quad v'' = 2$

$u^{(n-2)} = y^n$ $\quad v''' = 0$

$u = y'$	$v = x$
$u^n = y^{n+1}$	$v' = 1$
$u^{n-1} = y^n$	$v'' = 0$

$u = y$ $\quad v = 1$

$u^n = y^n$ $\quad v' = 0$

$y^n = u^nv + nu^{(n-1)}v' + \frac{n(n-1)}{2!}u^{(n-2)}v''$

$y^n = y^{n+2}x^2 + ny^{(n+1)}\cdot 2x + \frac{n(n-1)y^n}{2}\cdot 2 + y^{n+1}\cdot x + ny^n\cdot 1 + y^n\cdot 1$

$y^n = x^2y^{(n+2)} + 2xny^{(n+1)} + n(n-1)y^n + xy^{(n+1)} + ny^n + y^n$

$y^n = x^2y^{(n+2)} + 2nxy^{(n+1)} + xy^{(n+1)} + (n^2-n)y^n + ny^n + y^n$

$y^n = x^2y^{n+2} + (2n+1)xy^{(n+1)} + (n^2-n) + (n+1)y^n$

$y^n = x^2y^{(n+2)} + (2n+1)xy^{(n+1)} + (n^2+1)y^n$

$x^2y^{(n+2)} + (2n+1)xy^{(n+1)} + (n^2+1)y^n = 0.$