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Derive the Bernoulli Equation from the Newton's second law

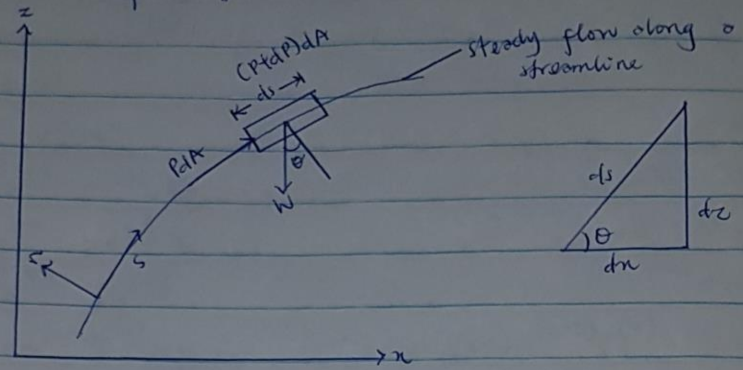

FIG 1.

Assumptions used in the derivation:

- Inviscid
- Incompressible
- Steady
- Conservative body force

According to Newton's second law of motion, $F = ma$

The component of Newton's second law along the streamline direction, s, can be written as

$$\sum \delta F_s = \delta m a_s$$
$$= \delta m V \frac{\delta v}{\delta s}$$
$$= \rho \delta \forall V \frac{\delta v}{\delta s}$$

Recall
$$a = \frac{dv}{dt}$$
$$a = \frac{dv}{ds} \cdot \frac{ds}{dt}$$
$$a = \frac{dv}{ds} \cdot V$$

The component of the weight force in the direction of the streamline:

$$\delta W_s = -\delta W \sin\theta = \gamma \delta \forall \sin\theta$$

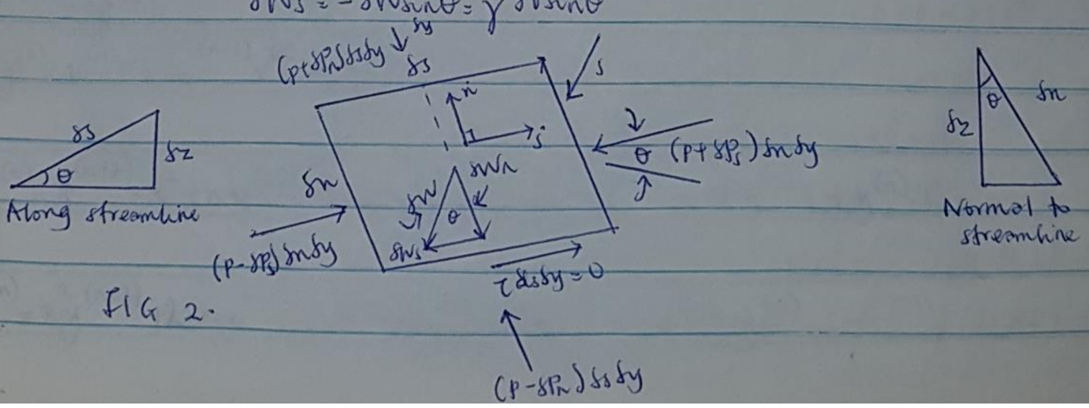

FIG 2.

The diagram FIG 2.8 the free body diagram of a fluid particle for which the important forces are due to pressure and gravity

1st-order Taylor series expansion for the pressure field:

$$\delta P_s \approx \frac{\delta P}{\delta s}\cdot\frac{\delta s}{2}$$

The net pressure force on the particle in the streamline direction.

$$\delta F_{ps} = (p-\delta P_s)\,\delta n\,\delta y - (p+\delta P_s)\,\delta n\,\delta y$$

$$= -2\delta P_s\,\delta n\,\delta y$$

$$= -\frac{\delta P}{\delta s}\,\delta s\,\delta n\,\delta y = -\frac{\delta P}{\delta s}\,\delta \forall$$

The net force acting in the streamline direction on the particle is

$$\sum \delta F_s = \delta W_s + \delta F_{ps} = \left(-\gamma \sin\theta - \frac{\delta P}{\delta s}\right)\delta \forall$$

$$-\gamma \sin\theta - \frac{\delta P}{\delta s} = \rho V \frac{\delta V}{\delta s} = \rho a_s$$

Note that $\sin\theta = \frac{\delta z}{\delta s}$ and $V\frac{\delta V}{\delta s} = \frac{1}{2}\frac{\delta V^2}{\delta s}$ along the streamline

The above equation can be integrated and rearranged:

$$-\gamma\frac{dz}{ds} - \frac{\delta P}{\delta s} = \frac{1}{2}\rho\frac{\delta V^2}{\delta s}$$

$$dP + \frac{1}{2}\rho d(V^2) + \gamma dz = 0 \quad \text{(along a streamline)}$$

For constant acceleration of gravity

$$\int\frac{dP}{\rho} + \frac{1}{2}V^2 + gz = C \quad \text{(along a streamline)}$$

For steady, inviscid and incompressible flow, we have the Bernoulli equation

$$P + \frac{1}{2}\rho V^2 + \gamma z = C$$

$F = ma$ Normal to a streamline

$$\sum \delta F_n = \delta m a_n = \frac{\delta m V^2}{R} = \rho\frac{\delta \forall V^2}{R}$$

The component of the weight (gravity force) in the normal direction:

$$\delta W_n = -\delta W \cos\theta = -\gamma\,\delta\forall \cos\theta$$

1st-order Taylor series expansion for the pressure field:

$$\delta P_n \approx \frac{\delta P}{\delta n}\,\frac{\delta n}{2}$$

The net pressure force on the particle in the streamline normal direction

$$\delta F_{pn} = (P-\delta P_n)\,\delta s\,\delta y - (P+\delta P_n)\,\delta s\,\delta y = -2\delta P_n\,\delta s\,\delta y$$

$$\delta F_{pn} = -\frac{\delta P}{\delta n}\,\delta s\,\delta n\,\delta y$$
$$= -\frac{\delta P}{\delta n}\,\delta \forall$$

The net force acting in the normal direction on the particle is

$$\sum \delta F_n = \delta W_n + \delta F_{pn}$$
$$= \left(-\gamma\cos\theta - \frac{\delta P}{\delta n}\right)\delta \forall$$

Note that $\cos\theta = \frac{\delta z}{\delta n}$, we obtain the equation of motion along the normal direction:

$$-\gamma\frac{\delta z}{\delta n} - \frac{\delta P}{\delta n} = \frac{\rho V^2}{R}$$

Since $\frac{\delta P}{\delta n} = \frac{dP}{dn}$ if s is constant, integrate across the streamline:

$$\int \frac{dP}{\rho} + \int \frac{V^2}{R}\,dn + gz = C \quad \text{(across the streamline)}$$

For steady, inviscid and incompressible flow, we have:

$$P + \rho\int \frac{V^2}{R}\,dn + \gamma z = C \quad \text{(across the streamline)}$$

It can also be expressed in this form

$$\frac{P}{\gamma} + \frac{V^2}{2g} + z = C \quad \text{(along a streamline)}$$

Pressure head: $\frac{P}{\gamma}$

Velocity head: $\frac{V^2}{2g}$

Elevation head: z

The Bernoulli equation states that the sum of the pressure head, the velocity head and the elevation head is constant along a streamline.