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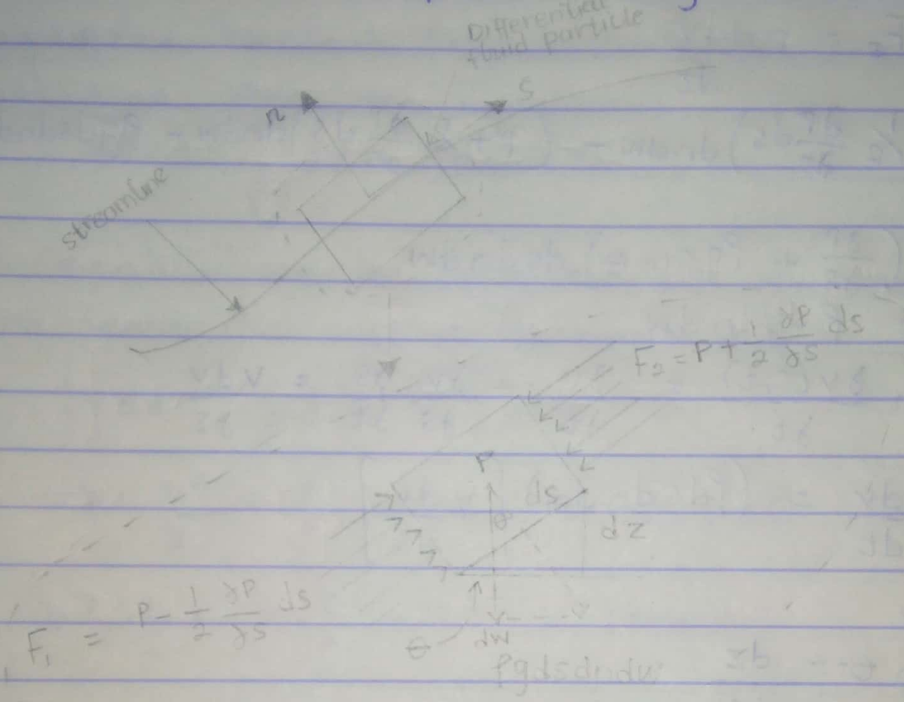
CHEMICAL ENGINEERING
CHE311

Derive Bernoulli's equation using Newton's second law.



A local coordinate system is shown in the figure. The s-direction is tangent to the streamline. The n-direction is normal to the streamline and the W direction is perpendicular to the streamline.

There is a pressure gradient dP/ds, along the direction of the streamline. There is also pressure gradient dP/dn, normal to the streamline if the streamline is curved. The Bernoulli equation is only concerned with the pressure gradient in the S-direction. The pressure at each end of the fluid element can be written in terms of the pressure at the centre of the fluid element and the S-direction pressure gradient.

Newton's Second Law states that the sum of the forces acting on a particle equals the particle's mass times its acceleration

Mathematically, this is written as

$$\sum \vec{F} = m\vec{a}$$

Newton's second law applied to the s-direction

$$\sum F_s = ma_s$$

$$\sum F_s = m\frac{dv}{dt}$$

$$= \left(p - \frac{1}{2}\frac{\partial p}{\partial s}ds\right)dndw - \left(p + \frac{1}{2}\frac{\partial p}{\partial s}ds\right)dndw - \rho g\,ds\,dn\,dw\sin\theta$$

$$= -\left(\frac{\partial p}{\partial s} + \rho g\sin\theta\right)ds\,dn\,dw$$

$$m = \rho\,ds\,dn\,dw$$

$$\frac{dv}{dt} = \frac{\partial v(s,t)}{\partial t} = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial s}\frac{\partial s}{\partial t} = v\frac{dv}{\partial s}$$

$$m\frac{dv}{dt} = (\rho\,ds\,dn\,dw)\left[v\frac{dv}{ds}\right]$$

Since

$$\sin\theta = \frac{dz}{ds}$$

$$\rho v\frac{dv}{ds} = -\frac{dP}{ds} - \rho g\frac{dz}{ds}$$

$$\rho v\frac{dv}{ds} + \frac{dp}{ds} + \rho g\frac{dz}{ds} = 0$$

$$v\frac{dv}{ds} + \frac{1}{\rho}\frac{dp}{ds} + g\frac{dz}{ds} = 0$$

$$v\,dv + \frac{dP}{\rho} + g\,dz = 0$$

The acceleration in the s-direction is the time rate of change of the velocity in the s-direction. The velocity also depends on where the particle is along the streamline. Hence, the velocity is a function of s and time, t. This allows the acceleration component in the s-direction to be written as

$$a_s = \frac{dv(s,t)}{dt} = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial s}\frac{\partial s}{\partial t} = \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial s}$$

Limiting the derivation to steady-state condition allows the S-direction acceleration to be written as;

$$a_s = V \frac{\partial v}{\partial s}$$

Applying the Newton's second law in the S-direction and simplifying gives the equation

$$V dv + \frac{1}{\rho} dP + g dz = 0$$

This equation can be integrated with an indefinite integral if the density is constant (incompressible).

$$\int v dv + \frac{1}{\rho} \int dP + \int g dz = 0$$

$$\Rightarrow \frac{v^2}{2} + \frac{P}{\rho} + gz = \text{constant}$$

$$\Rightarrow \frac{1}{2} v^2 + \frac{P}{\rho} + gz = \text{constant}$$

Bernoulli's equation states that the sum of the three terms shown in the equation is the same at any point along the length of the streamline.