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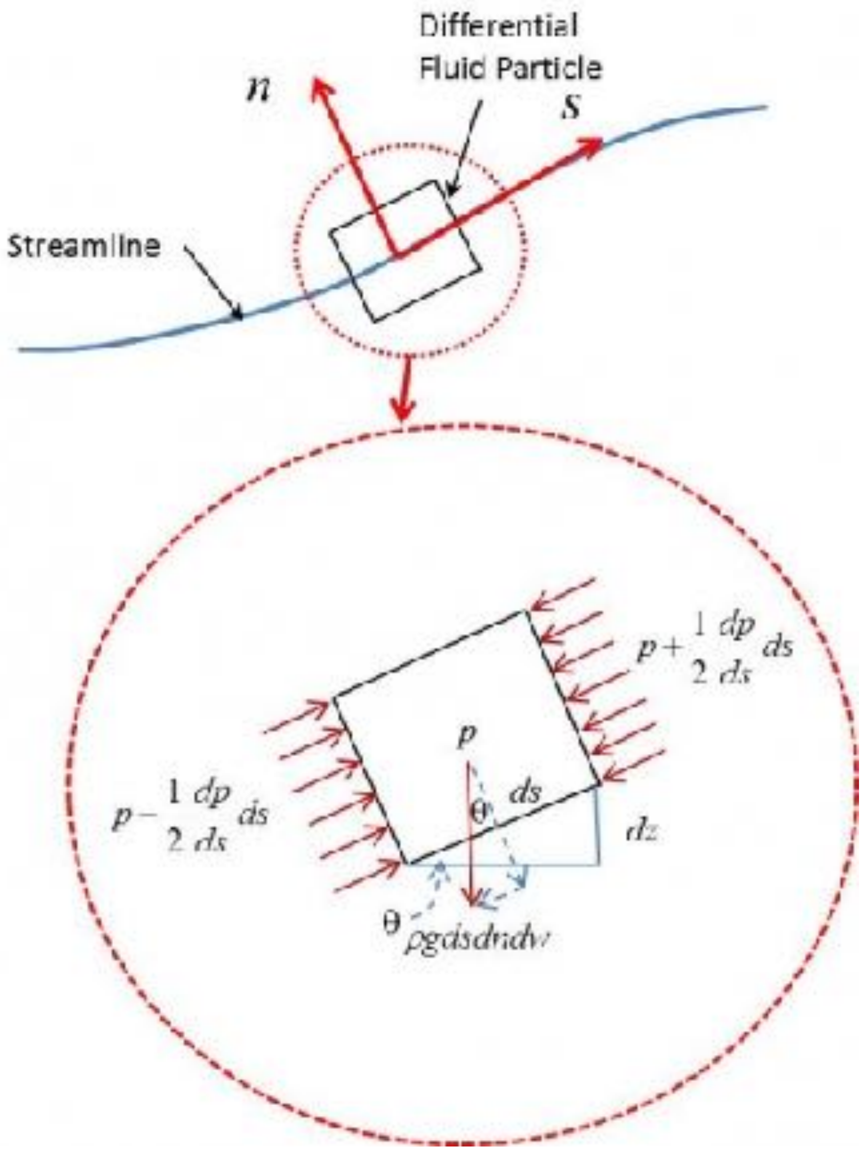
CHEMICAL ENGINEERING

CHE 311 - TRANSPORT PHENOMENA I : FLUID FLOW

Deriving Bernoulli's equation from Newton's second law



Consider a small particle of fluid that follows a streamline. A streamline is a line drawn through the flow field that is everywhere tangent to the velocity. The forces acting on the fluid particle are gravity and pressure. The viscous shear stresses are not included, so the fluid must be inviscid (negligibly small viscosity).

A local coordinate system is shown in the figure. The s-direction is tangent to the streamline. The n-direction is normal to the streamline, and the w-direction is perpendicular to and coming out of the page.

There is a pressure gradient, dP/ds, along the direction of the stream line. There is also a pressure gradient, dP/dn, normal to the streamline if the streamline is curved. The Bernoulli equation is only concerned with the pressure gradient in the s-direction.
The pressure at each end of the fluid element can be written in terms of the pressure at the center of the fluid element and the s-direction pressure gradient.

Note that the assumption used in the figure is that a positive pressure gradient exists in the positive s-direction.

Newton's Second Law states that the sum of the forces acting on a particle equals the particle's mass times its acceleration. Since we are tracking a particular fluid particle, Newton's second law can be applied directly. Mathematically, this is written as

$$F = ma$$

Newton's Second Law applied to the s-direction

$$F_s = ma_s$$

Step by Step Derivation

$$\sum F_s = m\frac{dV}{dt}$$

$$= \left(p - \frac{1}{2}\frac{dp}{ds}ds\right)dndw - \left(p + \frac{1}{2}\frac{dp}{ds}ds\right)dndw - \rho g ds dn dw \sin\theta$$

$$= -\left(\frac{dp}{ds} + \rho g \sin\theta\right)dsdndw$$

$$m = \rho dsdndw$$

$$\frac{dV}{dt} = \frac{\partial V(s,t)}{\partial t} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial s}\frac{\partial s}{\partial t} = V\frac{dV}{ds}$$

$$m\frac{dV}{dt} = (\rho dsdndw)\left(V\frac{dV}{ds}\right)$$

$$\sin\theta = \frac{dz}{ds}$$

$$\rho V\frac{dV}{ds} = -\frac{dp}{ds} - \rho g\frac{dV}{ds}$$

$$\rho V\frac{dV}{ds} + \frac{dp}{ds} + \rho g\frac{dz}{ds} = 0$$

$$V\frac{dV}{ds} + \frac{1}{\rho}\frac{dp}{ds} + g\frac{dz}{ds} = 0$$

$$VdV + \frac{dP}{\rho} + gdz = 0$$

The acceleration in the s-direction is the time rate of change of the velocity in the s-direction. The velocity also depends on where the particle is along the streamline. Hence, the velocity is a function of s and time, t. This allows the acceleration component in the s-direction to be written as

$$a_s = \frac{dV(s,t)}{dt} = \frac{\delta V}{\delta t} + \frac{\delta V}{\delta s}\frac{\delta s}{\delta t} = \frac{\delta V}{\delta t} + v\frac{\delta V}{\delta s}$$

$$a_s = V\frac{dV}{ds}$$

Applying Newton's Second Law in the s-direction and simplifying yields the equation

$$VdV + \frac{1}{\rho}dp + gdz = 0$$

This equation can be integrated with an indefinite integral if the density is constant (incompressible).

$$\frac{1}{2}V^2 + \frac{p}{\rho} + gz = constant$$

v is the fluid flow speed at a point on a streamline,

g is the acceleration due to gravity,

z is the elevation of the point above a reference plane, with the positive z-direction pointing upward – so in the direction opposite to the gravitational acceleration,

p is the pressure at the chosen point, and

ρ is the density of the fluid at all points in the fluid.

Multiplying through by g

$$\frac{V^2}{2g} + \frac{p}{\rho g} + z = constant$$

$$\frac{V_1^2}{2g} + \frac{p_1}{\rho g} + z_1 = \frac{V_2^2}{2g} + \frac{p_2}{\rho g} + z_2$$