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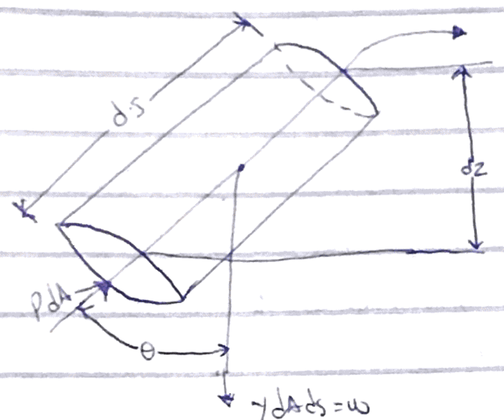
Derive the Bernoulli's equation from the Newton second law

Let us consider a frictionless steady flow along a streamline. A cylindrical differential element as shown in the figure below of



Length ds and cross sectional area dA is assumed along streamline intensity of pressures on the force of the element are P and (P + dp). So the pressure force acting on the element tending to acceleration fluid mass will be

$$P.dA - (P+dp)dA = -dp.dA \quad ---- (1)$$

If γ be the unit weight of fluid mass, gravity force acting vertically downward will be (γ.dA.ds). Component of gravity force along direction of motion will be

$$= -(\gamma.dA.ds)\cos\theta = -(\gamma.dA.ds)\frac{dz}{ds} = (\gamma.dAdz) \quad --- (2)$$

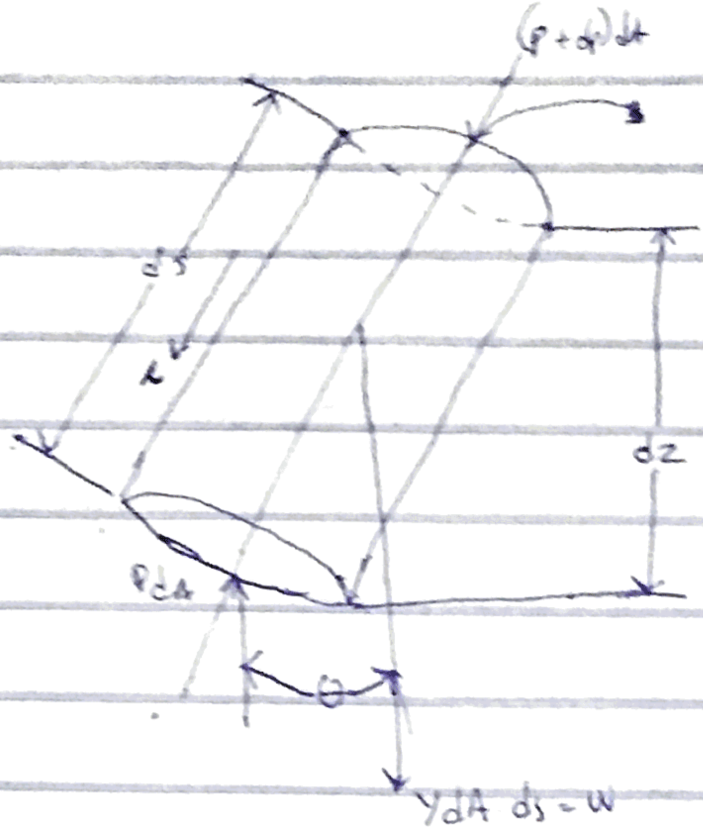
The velocity being V along the direction of motion acceleration for the steady flow will be $(V\, dv/ds)$. Applying Newton's Law of motion, we have

$$-dp.dA - \gamma.dA.dz = \gamma/g\, dA \cdot ds.\, V\, dv/ds$$

$$\Rightarrow dp/\rho + V\,dv + g\,dz = 0 \quad --- (3)$$

This equation is known as one-dimensional Euler's equation for an ideal fluid. It is applicable to both compressible and incompressible fluid flow

For a real fluid, in the figure below



An additional term, force due to fluid friction will be effective. If τ be the shear stress and r be the radius of the fluid element, the frictional force along the direction of motion will be $(-\tau \cdot 2 \cdot \pi r\, ds)$.
Modifying equation (3) we arrive at

$$-dp.dA - \gamma.dA.dz - 2\pi r.\tau.ds = \frac{\gamma}{g}\, dA.ds.V\frac{dv}{ds}$$

$$\Rightarrow \frac{dp}{\rho} + Vdv + gdz = \frac{-2\tau.ds}{\rho r} \quad \text{--- (4)}$$

This is Euler's equation for one dimensional real fluid flow

To derive Bernoulli's equation from Euler's equation for the case of an incompressible ideal fluid where ρ = Constant; equation (3) can be integrated as

$$\int \frac{dp}{\rho} + \int v\,dv + \int g\,dz = \text{Constant}$$

$$\Rightarrow \frac{P}{\rho} + \frac{V^2}{2} + gz = \text{Constant} \quad \text{--- (5)}$$

$$\Rightarrow \frac{P}{\gamma} + \frac{V^2}{2g} + Z = \text{Constant}$$

If the fluid becomes real and incompressible, integration over two section 1 and 2 of total fluid element length l yields

$$\int_1^2 \frac{dp}{\rho} + \int_1^2 v\,dv + \int_1^2 g\,dz = \int_1^2 \frac{2\tau.ds}{\rho r}$$

$$\Rightarrow \frac{P_2 - P_1}{\rho} + \frac{V_2^2 - V_1^2}{2} + g(z_2 - z_1) = \frac{-2\tau}{\rho r}(l_2 - l_1)$$

$$\Rightarrow \frac{P_2 - P_1}{\gamma} + \frac{V_2^2 - V_1^2}{2g} + (z_2 - z_1) = \frac{-2\tau l}{g r} \quad \text{--- (6)}$$

$$\Rightarrow \left(\frac{P_1}{\gamma} + \frac{V_1^2}{2g} + Z_1\right) - \left(\frac{P_2}{\gamma} + \frac{V_2^2}{2g} + Z_2\right) = \frac{-2\tau l}{\gamma r}$$

applying equation (4.11) over two sections 1 and 2 we have

$$\left(\frac{P_1}{\gamma} + \frac{V_1^2}{2g} + Z_1\right) - \left(\frac{P_2}{\gamma} + \frac{V_2^2}{2g} + Z_2\right) = 0$$

Here right-hand side of equation (4.12) is designated as

$$h_f = \frac{2\tau l}{\gamma \cdot r} \quad \text{----} \ (7)$$

This is the head-loss between two pre-selected sections in a conduit carrying real fluid. Equation (5) and (6) are popularly known as Bernoulli's equation for real and ideal fluid respectively