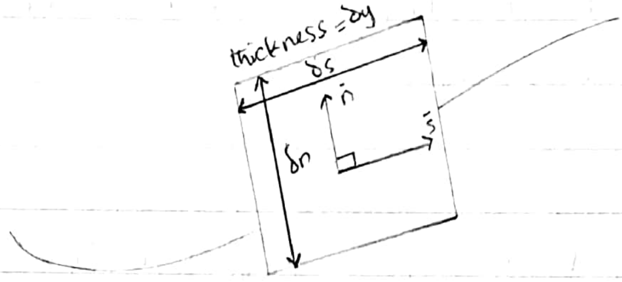


Fig 1:

$\forall = \delta s \times \delta n \times \delta y$

From <u>Newtons second law</u> which states that the sum of all forces ac in the streamline direction of the element

$$\Sigma \delta F_s = \delta m a_s \quad \text{———} (1)$$

Using chain rule, acceleration can be re-written as

$$a_s = \frac{dv}{dt} = \left(\frac{dv}{ds}\right)\left(\frac{ds}{dt}\right)$$

$\frac{ds}{dt}$ is the velocity in the streamline direction

$$a_s = V \frac{dv}{ds}$$

Newton's second law will now be

$$\Sigma \delta F_s = \delta m \, V \frac{\delta v}{\delta s} \quad \text{———} (2)$$

Pulling the density out of the mass term, we have

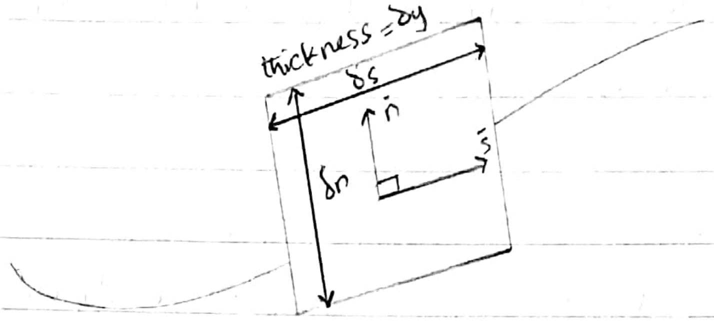
$$\Sigma \delta F_s = \rho \delta \forall \, V \frac{\delta v}{\delta s} \quad \text{———} (3)$$

This equation is valid for compressible and incompressible fluids.

Derive the Bernoulli Equation from the Newton Second Law.

Solution

Considering a fluid element following a stream line.



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$$\Sigma \delta F_s = \delta W_s + \delta F_{ps}$$

$$= -\gamma \delta \forall \sin\theta + \frac{\partial p}{\partial s} \gamma$$

$$= \left(-\gamma \sin\theta - \frac{\partial p}{\partial s}\right)\delta\forall$$

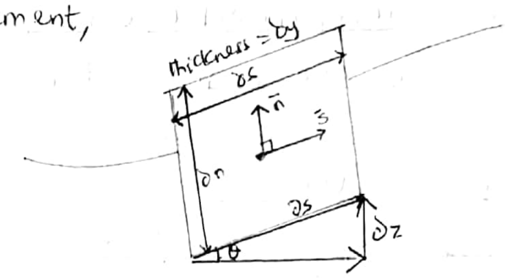
recall $\Sigma \delta F_s = \rho \delta\forall \cdot V \frac{\partial V}{\partial s}$

$$\rho \cancel{\delta\forall}\, V \frac{\partial V}{\partial s} = \left(-\gamma \sin\theta - \frac{\partial p}{\partial s}\right)\cancel{\delta\forall}$$

$$\rho V \frac{\partial V}{\partial s} + \gamma \sin\theta + \frac{\partial p}{\partial s} = 0$$

$$\frac{\partial p}{\partial s} + \rho V \frac{\partial V}{\partial s} + \gamma \sin\theta = 0 \quad \text{——— (6)}$$

From fluid element,



$$\sin\theta = \frac{dz}{ds}$$

Putting $\sin\theta$ into —(6)

$$\frac{\partial p}{\partial s} + \rho V \frac{\partial V}{\partial s} + \gamma \frac{dz}{ds} = 0$$

$$dp = \frac{\partial p}{\partial s} ds + \frac{\partial p}{\partial n} dn$$

when $dn = 0$

$$dp = \frac{\partial p}{\partial s} ds$$

$$\frac{dp}{ds} = \frac{\partial p}{\partial s}$$

Puttin $\frac{\partial p}{\partial s}$ into —(6)

$$\frac{dp}{ds} + \rho V \frac{\partial V}{\partial s} + \gamma \frac{dz}{ds}$$

Assuming the fluid is incompressible

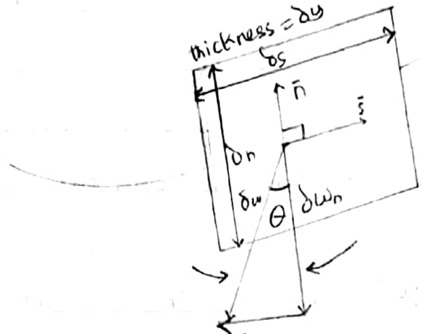


Fig 2

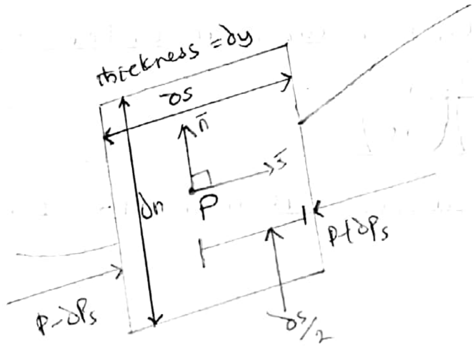
We need to re-write -③ (LHS), the sum of all the forces acting on the part by collecting all the forces that are acting in the streamline direction. First force is gravity which gives the particle a weight.

$\delta w = -\gamma \delta \forall \sin\theta$ ——④

the negative sign appears because of the way we've drawn our axis in Fig: From the Fig 2 the streamline is horizontal i.e $\theta = 0$, the weight component the streamline direction is also zero

$\therefore$ the weight term $\delta w = -\gamma \delta \forall \sin\theta$

As pressure acts on the fluid,



Pressure is not constant throughout the fluid because of its weight. So we can say pressure $P(s,n)$

$$\delta P_s = \frac{\partial P}{\partial s}\frac{\delta s}{2}$$

Total force acting on the fluid element

$$\delta F_{ps} = (P-\delta P_s)\delta n\delta y - (P+\delta P_s)\delta n\delta y$$

$$= -2\delta P_s \delta n \delta y \qquad = \frac{\partial P}{\delta s}\delta \quad ⑤$$

Putting the value of $V\frac{dv}{ds}$ in eq — (6)

$$\frac{dP}{ds} + \frac{1}{2\rho}\frac{d(v^2)}{ds} + \gamma\frac{dz}{ds} = 0$$

$$dp + \frac{1}{2\rho}d(v^2) + \gamma dz = 0 \quad \text{——— (7)}$$

Integrating

$$\int\frac{dp}{\rho} + \frac{1}{2}v^2 + gz = \text{Constant} \quad \text{——— (8)}$$

Since we assumed the fluid is incompressible, density and specific weight will remain constant and this can be re-written as

$$P + \frac{1}{2}\rho v^2 + \gamma z = \text{Constant.} \quad \text{— (9)}$$

This (9) gives the Bernoulli Equation.