

Ayeni Ajobami Opeyemi

15/ENG04/011

Assignment 4

$$1. (1-x^2) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + 2y = 0$$

$$(1-x^2)y^{(n)} - 2xy^{(n)} + 2y = 0$$

$$y^{(n)} = U^n V + n U^{(n-1)} V^{(1)} + \frac{n(n-1)}{2!} U^{(n-2)} V^{(2)} + \dots$$

$$\left[y^{(2+n)} \cdot (1-x^2) + n y^{(1+n)} \cdot (-2x) + \frac{n(n-1)}{2!} y^{(n)} \cdot (-2) \right] + \left[y^{(1+n)} \cdot (-2x) + n y^{(n)} \cdot (-2) \right] + \left[2y^{(n)} \right] = 0$$

$$(1-x^2)y^{(2+n)} - 2xny^{(1+n)} - n(n-1)y^{(n)} - 2xy^{(1+n)} - 2ny^{(n)} + 2y^{(n)} = 0$$

let $x=0$

$$y^{(2+n)} - n(n-1)y^{(n)} - 2ny^{(n)} + 2y^{(n)} = 0$$

$$y^{(2+n)} + y^{(n)} [-n(n-1) - 2n + 2] = 0$$

$$y^{(2+n)} + y^{(n)} [-n^2 + n - 2n + 2] = 0$$

$$y^{(2+n)} + y^{(n)} [-n^2 - n + 2] = 0$$

$$\boxed{y^{(2+n)} = -y^{(n)} [-n^2 - n + 2]} \quad \text{recurrence relation}$$

$$n=0 \quad \therefore (y^{(0)})_0 = -y^{(0)}_0 \cdot (2) = -2(y^{(0)})_0$$

$$n=1 \quad \therefore (y^{(1)})_0 = -y^{(1)}_0 \cdot [0] = 0$$

$$n=2 \quad \therefore (y^{(2)})_0 = -y^{(2)}_0 \cdot [-4] = 4(y^{(2)})_0 = (4)(-2)(y^{(0)})_0$$

$$n=3 \quad \therefore (y^{(3)})_0 = -y^{(3)}_0 \cdot [-10] = 10(y^{(3)})_0 = (10)(0) = 0$$

$$n=4 \quad \therefore (y^{(4)})_0 = -y^{(4)}_0 \cdot [-18] = 18(y^{(4)})_0 = (18)(4)(-2)(y^{(0)})_0$$

$$n=5 \quad \therefore (y^{(5)})_0 = -y^{(5)}_0 \cdot [-28] = 28(y^{(5)})_0 = (28)(0) = 0$$

$$y = (y^0)_0 + x(y^{(1)})_0 + \frac{x^2}{2!} (y^{(2)})_0 + \frac{x^3}{3!} (y^{(3)})_0 + \dots$$

$$y = (y^0)_0 + x(y^{(1)})_0 + \frac{x^2}{2!} (-2)(y^{(0)})_0 + \frac{x^3}{3!} (0) + \frac{x^4}{4!} (4)(-2)(y^{(0)})_0 + \dots$$

$$\dots + \frac{x^6}{6!} (0) + \frac{x^6}{6!} (18)(4)(-2)(y^{(0)})_0 + \frac{x^8}{8!} (0)$$

$$y = (y^0)_0 + x(y^{(1)})_0 - \frac{x^2}{2} (y^{(0)})_0 - \frac{2^4}{3 \times 2} (y^{(0)})_0 - \frac{2^6}{5} (y^{(0)})_0$$

$$y = (y^0)_0 \left[1 - x^2 - \frac{x^4}{3} - \frac{x^6}{5} \dots \right] + (y^{(1)})_0 [2x \dots]$$

$$2. \textcircled{i} \quad 3e^{-4t} - 5e^{4t} = f(t)$$

$$L[f(t)] = \frac{3}{s+4} - \frac{5}{s-4} = \frac{3(s-4) - 5(s+4)}{(s+4)(s-4)} = \frac{3s-12-5s-20}{(s+4)(s-4)} \\ = \frac{-2s-32}{(s+4)(s-4)}$$

$$\textcircled{ii} \quad \sin 4t + \cos 4t = f(t)$$

$$L[f(t)] = \frac{4}{s^2+4^2} + \frac{s}{s^2+4^2} = \frac{4+s}{(s^2+4^2)}$$

$$\textcircled{iii} \quad t^3 + 2t^2 - 6t + 4$$

$$\frac{n!}{s^{n+1}} = \frac{3!}{s^4} + 2 \cdot \frac{2!}{s^3} - \frac{1!}{s^2} + \frac{4}{s} = \frac{6}{s^4} + \frac{4}{s^3} - \frac{1}{s^2} + \frac{4}{s}$$

$$= \frac{1}{s^4} [6 + 4s - s^2 + 4s^3]$$

$$\textcircled{iv} \quad t \sin 3t = f(t)$$

$$L[\sin 3t] = \frac{3}{s^2+3^2} = \frac{3}{s^2+9}$$

$$L[t \sin 3t] = -\frac{d}{ds} F(s)$$

$$u = 3 \quad v = s^2 + 9 \quad \therefore \frac{(s^2+9) \cdot 0 - 3(2s)}{(s^2+9)^2} = \frac{-6s}{(s^2+9)^2} = \frac{6s}{(s^2+9)^2}$$

$$\textcircled{v} \quad e^{-2t} \cos 5t = f(t)$$

$$L[f(t)] = L[\cos 5t] = \frac{s}{s^2+5^2}$$

$$L[f(t)] = \frac{s+2}{(s+2)^2 + 5^2}$$

$$\textcircled{vi} \quad \frac{e^{-t} - e^{-2t}}{t}$$

$$\lim_{t \rightarrow 0} \left[\frac{-e^{-t} + 2e^{-2t}}{1} \right] = \frac{2}{1} = 2$$

$$L\left[\frac{e^{-t} - e^{-2t}}{t}\right] = \frac{1}{s+1} - \frac{1}{s+2} = f(s) = \frac{s+2 - 2(s+1)}{(s+1)(s+2)} \\ = \frac{s+2 - 2s - 2}{(s+1)(s+2)} = \frac{-s}{(s+1)(s+2)}$$

$$L\left[\frac{e^{-t} - e^{-2t}}{t}\right] = \int_{\sigma=2.5}^{\infty} \left(\frac{1}{\sigma+1}\right) - \left(\frac{1}{\sigma+2}\right) d\sigma$$

$$= \int_{\sigma=2.5}^{\infty} \frac{1}{\sigma+1} d\sigma - \int_{\sigma=2.5}^{\infty} \frac{1}{\sigma+2} d\sigma$$

$$= \left[\ln(\sigma+1) - \ln(\sigma+2) \right]_{\sigma}^{\infty}$$

$$= \left[\ln(\sigma+1) - \ln(\sigma+2) \right]_{\sigma}^{\infty}$$

$$= \left[\ln \frac{\sigma+1}{(\sigma+2)^{\circ}} \right]_{\sigma}^{\infty} = \ln \left[\frac{(\infty+1)^{\circ}}{(\infty+2)^{\circ}} - \frac{\sigma+1}{(\sigma+2)^{\circ}} \right]$$

$$= -\ln \left[\frac{\sigma+1}{(\sigma+2)^{\circ}} \right] = \ln \left[\frac{(\sigma+2)^{\circ}}{\sigma+1} \right] //$$

(vii) $e^{4t} \cos 2t$

$$L[\cos 2t] = \frac{s}{s^2+2^2} = \frac{s}{s^2+4}$$

$$L[e^{4t} \cos 2t] = \frac{s-4}{(s-4)^2+4}$$

(viii) $t \sin 2t$

$$L[\sin 2t] = \frac{2}{s^2+2^2}$$

$$L[t \sin 2t] = -\frac{d}{ds} \left[\frac{2}{s^2+4} \right]$$

$$u=2 \quad du=0; \quad v=s^2+4 \quad dv=2s$$

$$\frac{(s^2+4) \cdot 0 - 2(2s)}{(s^2+4)^2} = \left(\frac{-4s}{(s^2+4)^2} \right) = \frac{4s}{(s^2+4)^2}$$

$$1x. t^3 + 4t^2 + 5 = f(t)$$

$$= \frac{3!}{s^4} + \frac{4 \cdot 2!}{s^3} + \frac{5}{s} = \frac{6}{s^4} + \frac{8}{s^3} + \frac{5}{s}$$

$$= \frac{1}{s^4} [6 + 8s + 5s^3]$$

$$7x. e^{3t} (t^2 + 4)$$

$$L[t^2 + 4] = \frac{2!}{s^3} + \frac{4}{s} = \frac{2}{s^3} + \frac{4}{s}$$

$$L[e^{3t}] = \frac{2}{(s-3)^3} + \frac{4}{(s-3)} = \frac{4s^2 - 24s + 36}{(s-3)^3}$$

$$11x. t^2 \cos t = f(t)$$

$$L[\cos t] = \frac{s}{s^2+1} \quad \therefore L[t^2 \cos t] = -\frac{d^2}{ds^2} \left[\frac{s}{s^2+1} \right]$$

$$\frac{d}{ds} \left[\frac{s}{s^2+1} \right] = \frac{(s^2+1) \cdot 1 - s(2s)}{(s^2+1)^2} = \frac{s^2+1-2s^2}{(s^2+1)^2} = \frac{1-s^2}{(s^2+1)^2}$$

$$\frac{d^2}{ds^2} \left[\frac{1-s^2}{(s^2+1)^2} \right] \quad \therefore U = 1-s^2 \quad dU = -2s$$

$$V = (s^2+1)^2 \quad dV = 4s(s^2+1)$$

$$V = s^2+1 \quad dV = 2s$$

$$W = V^2 \quad dW = 2V \cdot dV$$

$$\frac{dV}{ds} \times \frac{dW}{dV} = 2s \times 2V$$

$$\frac{d^2}{ds^2} = 4sV = 4s(s^2+1)$$

$$\frac{(s^2+1)^2 \cdot (-2s) - (1-s^2) \cdot 4s(s^2+1)}{(s^2+1)^4} = \frac{-2s(s^2+1)^2 - 4s(s^2+1)(1-s^2)}{(s^2+1)^4}$$

$$= \frac{(s^2+1) \left[-2s(s^2+1) - 4s(1-s^2) \right]}{(s^2+1)^3} = \frac{-2s[s^2+1 - 2 + 2s^2]}{(s^2+1)^3}$$

$$= \frac{-2s[3s^2-1]}{(s^2+1)^3} \quad \therefore L[t^2 \cos t] = -\frac{d^2}{ds^2} \left[\frac{s}{s^2+1} \right]$$

$$\therefore \frac{d^2}{ds^2} \left[\frac{s}{s^2+1} \right] = \frac{2s[3s^2-1]}{(s^2+1)^3}$$

(XII) $\frac{\sinh 2t}{t} = f(s)$

$$\lim_{t \rightarrow 0} \left[\frac{\sinh 2t}{t} \right] = \frac{2 \cosh 2t}{1} = \frac{2}{1} = 2$$

$$L \left[\frac{\sinh 2t}{t} \right] = \int_0^{\infty} \frac{2}{s^2 - 2^2} ds = \frac{2}{s^2 - 4}$$

$$L \left[\frac{\sinh 2t}{t} \right] = \int_{0^+}^{\infty} \frac{2}{s^2 - 4} ds = 2 \int_{0^+}^{\infty} \frac{1}{s^2 - 4} ds$$

3 (i) $\frac{s-5}{(s-3)(s-4)} = f(s) = \frac{A}{s-3} + \frac{B}{s-4}$

A: $s-3 f(s) = s-3 \cdot \frac{s-5}{(s-3)(s-4)} \Big|_{s=3} = \frac{(3-5)}{(3-4)} = 2$

B: $s-4 f(s) = s-4 \cdot \frac{s-5}{(s-3)(s-4)} \Big|_{s=4} = \frac{4-5}{4-3} = -1$

$$f(s) = \frac{2}{s-3} - \frac{1}{s-4}$$

$$f(t) = \underline{\underline{2e^{3t} - e^{4t}}}$$

$$(i) \frac{2s-6}{(s-2)(s-4)} = f(s) = \frac{A}{s-2} + \frac{B}{s-4}$$

$$A: \frac{2s-6}{s-4} \Big|_{s=2} = \frac{2(2)-6}{2-4} = 1$$

$$B: \frac{2s-6}{s-2} \Big|_{s=4} = \frac{2(4)-6}{4-2} = 1$$

$$f(s) = \frac{1}{s-2} + \frac{1}{s-4}$$

$$f(t) = e^{2t} + e^{4t}$$

$$(ii) \frac{5s-8}{s(s-4)} = f(s) = \frac{A}{s} + \frac{B}{s-4}$$

$$A: \frac{5s-8}{s-4} \Big|_{s=0} = \frac{5(0)-8}{0-4} = \frac{8}{4} = 2$$

$$B: \frac{5s-8}{s} \Big|_{s=4} = \frac{5(4)-8}{4} = 3$$

$$f(s) = \frac{2}{s} + \frac{3}{s-4}$$

$$f(t) = 2 + 3e^{4t}$$

$$(iii) \frac{s^2-3s-4}{(s-3)(s-1)^2} = f(s) = \frac{A}{s-3} + \frac{B}{s-1} + \frac{C}{(s-1)^2}$$

$$A: \frac{s^2-3s-4}{(s-1)^2} \Big|_{s=3} = \frac{3^2-3(3)-4}{(3-1)^2} = -1$$

$$B: \frac{s^2-3s-4}{s-3} \Big|_{s=1} = \frac{1^2-3(1)-4}{1-3} = 3$$

$$C: \frac{d}{ds} \left[\frac{s^2-3s-4}{s-3} \right] \Big|_{s=1} = \frac{(s-3)(2s-3) - [s^2-3s-4]}{(s-3)^2}$$

$$\text{at } s=1 \quad \frac{(1-3)(2(1)-3) - [1^2-3(1)-4]}{(1-3)^2} = \frac{2}{4}$$

$$F(s) = \frac{-1}{s-3} + \frac{3}{(s-1)^2} + \frac{2}{s-1}$$

$$f(t) = -e^{3t} + 3te^t + 2e^t$$

$$= e^t [3t + 2] - e^{3t}$$

$$(V) \frac{s-5}{s^2+4s+20} = F(s)$$

$$F(s) = \frac{s-5}{s^2+4s+4+16} = \frac{s}{(s+2)^2+16} - \frac{5}{(s+2)^2+16}$$

$$= \frac{s+2-2}{(s+2)^2+16} - \frac{5}{(s+2)^2+16} = \frac{s+2}{(s+2)^2+16} - \frac{7}{(s+2)^2+16}$$

$$= \frac{s+2}{(s+2)^2+4^2} - \frac{7 \times \frac{4}{4}}{(s+2)^2+4^2}$$

$$= \frac{s+2}{(s+2)^2+4^2} - \frac{7}{4} \cdot \frac{4}{(s+2)^2+4^2}$$

$$f(t) = e^{-2t} \cos 4t - \frac{7}{4} e^{-2t} \sin 4t$$

$$= e^{-2t} \left[\cos 4t - \frac{7}{4} \sin 4t \right]$$