Alamin Mohammed Nabil

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1 $(1-x^2)\, d^2y/dx^2 - 2x\, dy/dx + 2y = 0$

$(1-x^2)y'' - 2xy' + 2y = 0$

let $(1-x^2)y'' = W$

$V = 1-x^2 \quad V' = -2x \quad V^2 = -2 \quad V^3 = 0$

$U = y'' \quad U^n = y^{n+2}$

$W^n = U^n V + nU^{n-1}V' + \frac{n(n-1)U^{n-2}V^2}{2} + 0$

$= y^{n+2}(1-x^2) + ny^{n-1}(-2x) + n(n-1)y^n$

$= (1-x^2)y^{n+2} + -2xny^{n-1} + n(n-1)y^n$

let $W = -2xy'$

$V = -2x \quad V' = -2 \quad V^2 = 0$

$U = y' \quad U^n = y^{n+1}$

$W^n = U^n V + nU^{n-1}V' + 0$

$= y^{n+1}(-2x) + ny^n(-2)$

$= -2xy^{n+1} - 2ny^n$

let $W = 2y$

$V = 2 \qquad V' = 0$

$U = y \qquad U^n = y^n$

$W^n = U^n V_0 + 0$

$= 2y^n$

$W = (1 - x^2) y^{n+2} - 2xn y^{n+1} + n(n-1) y^n + \cancel{2x}$

$- 2x y^{n+1} - 2n y^n + 2y^n$

$= (1-x^2) y^{n+2} + 2x y^{n+1} (-n-1) + y^n (n^2 - n - 2n + n)$

$= (1-x^2) y^{n+2} + 2x y^{n+1} (-n-1) + y^n (n^2 - 3n + n)$

at $x = 0$

$= y^{n+2} + y^n (n^2 - 3n + n)$

recurrence relation ; $(y^{n+2})_0 = (-y^n (n^2 - 3n + n))_0$

$n = 0$

$(y^2)_0 = 0$

$n = 1$

$(y^3)_0 = +(y')_0$

$n = 2$

$(y^4)_0 = \cancel{0}\ 0$

$n = 3$

$(y^5)_0 = (-3y^3)_0 = -3(y'_0)$

$n = 4$

$(y^6)_0 = 0$

$n = 5$

$(y^7)_0 = -15y^5 = -15(-3y'_0)$

$n = 6$

$(y^8)_0 = 0$

$$y = y_0 + x y'_0 + \frac{x^2}{2!} y^2_0 + \frac{x^3}{3!} y^3_0 +$$

$$\frac{x^4}{4!} y^4_0 + \frac{x^5}{5!} y^5_0 + \frac{x^6}{6!} y^6_0 + \frac{x^7}{7!} y^7_0 + \frac{x^8}{8!} y^8_0$$

$$y = y_0 + x y'_0 + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(y'_0) + \frac{x^4}{4!}(0) +$$

$$\frac{x^5}{5!}(-3y'_0) + 0 + \frac{x^7}{7!}(-15(-3y'_0)) + 0$$

$$y = y_0 + x y'_0 + \frac{x^3 y'_0}{6} - \frac{y'_0 x^5}{40} + \frac{3y'_0 x^7}{336}$$

$$\cancel{y = y_0 (1 + x^7 \ldots) +}$$

$$y = y_0 + y'_0 \left[x + \frac{x^3}{6} - \frac{x^5}{40} + \frac{x^7}{112} + \ldots \right]$$

i) $L[3e^{-4t} - 5e^{4t}] = \frac{3}{s+4} - \frac{5}{s-4}$

ii) $L[\sin 4t + \cos 4t] = \frac{4}{s^2+4^2} + \frac{s}{s^2+4^2} = \frac{4+s}{s^2+16}$

iii) $L[t^3 + 2t^2 - t + 4]$

$= \frac{3!}{s^{3+1}} + \frac{2(2!)}{s^{2+1}} - \frac{1!}{s^{1+1}} + \frac{4}{s}$

$= \frac{6}{s^4} + \frac{4}{s^3} - \frac{1}{s^2} + \frac{4}{s}$

iv) $e^{-2t} \cos 5t$

$L[\cos 5t] = \frac{s}{s^2+25}$

$L[e^{-2t} \cos 5t] = \frac{(s+2)}{(s+2)^2 + 25}$

v) $t \sin 3t$

$L[\sin 3t] = \frac{3}{s^2+9}$

$L[t \sin 3t] = \frac{-d}{ds}\left[\frac{3}{s^2+9}\right]$

$\frac{3}{s^2+9} = 3(s^2+9)^{-1}$

$u = s^2+9 \qquad \therefore y = 3u^{-1}$

$\frac{du}{ds} = 2s \qquad \frac{dy}{du} = -3u^{-2}$

$\frac{dy}{ds} = -6su^{-2} = -6s(s^2+9)^{-2} = \frac{-6s}{(s^2+9)^2}$

$Vii)$ $e^{1t}\cos 2t$

$L[\cos 2t] = \frac{s}{s^2+4}$

$L[e^{1t}\cos 2t] = \frac{(s-1)}{(s-1)^2+4}$

$Viii)$ $t\sin 2t$

$L[\sin 2t] = \frac{2}{s^2+4}$

$L[t\sin 2t] = \frac{-d}{ds}\left(\frac{2}{s^2+4}\right)$

$2(s^2+4)^{-1}$

$u = s^2+4$ $\quad \therefore y = 2u^{-1}$ $\quad \frac{dy}{ds} = -4su^{-2} = \frac{-4s}{u^2}$

$\frac{du}{ds} = 2s$ $\quad \frac{dy}{du} = -2u^{-2}$

$= \frac{-4s}{(s^2+4)^2}$

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$\therefore L[t \sin 2t] = \dfrac{4s}{(s^2+4)^2}$

ix) $t^3 + 4t^2 + 5$

$L[t^3 + 4t^2 + 5] = \dfrac{3!}{s^{3+1}} + \dfrac{4(2!)}{s^{2+1}} + \dfrac{5}{s}$

$= \dfrac{6}{s^4} + \dfrac{8}{s^3} + \dfrac{5}{s}$

x) $e^{3t}(t^2+4)$

$L[t^2+4] = \dfrac{2!}{s^{2+1}} + \dfrac{4}{s}$

$= \dfrac{2}{s^3} + \dfrac{4}{s}$

$L[e^{3t}(t^2+4)] = \dfrac{2}{(s-3)^3} + \dfrac{4}{(s-3)^3}$

xi) $t^2 \cos t$

$L[\cos t] = \dfrac{s}{s^2+1}$

$L[t^2 \cos t] = \dfrac{d^2}{ds^2}\left(\dfrac{s}{s^2+1}\right)$

$u = s \qquad v = s^2+1$

$\dfrac{du}{ds} = 1 \qquad \dfrac{dv}{ds} = 2s$

$\dfrac{d}{ds} \dfrac{(s^2+1)1 - 2s^2}{(s^2+1)^2} = \dfrac{1-s^2}{(s^2+1)^2}$

$2s$ $4su$ $u^2 = 2u$

$\frac{d^2}{ds^2} =$

$u = 1 - s^2$ $\qquad$ $v = (s^2+1)^2$

$\frac{du}{ds} = -2s$ $\qquad$ $\frac{dv}{ds} = 4s(s^2+1)$

$$\frac{(s^2+1)^2\left(-2s\right) - (1-s^2)\left(4s(s^2+1)\right)}{\left((s^2+1)^2\right)^2}$$

$$= \frac{-2s(s^2+1)^2 - 4s(s^2+1)(1-s^2)}{(s^2+1)^4}$$

$$L[t^2 \cos t] = \frac{-2s(s^2+1)^2 - 4s(s^2+1)(1-s^2)}{(s^2+1)^4}$$

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vi)

$$L\left[\frac{e^{-t} - e^{-2t}}{t}\right]$$

$$L\left[e^{-t} - e^{-2t}\right]$$

$$= \frac{1}{s+1} - \frac{1}{s+2}$$

$$L\left[\frac{e^{-t} - e^{-2t}}{t}\right] = \int_{\sigma=s}^{\infty}\left(\frac{1}{\sigma+1} - \frac{1}{\sigma+2}\right)d\sigma$$

$$= \Big[\ln(\sigma+1) - \ln(\sigma+2)\Big]_{\sigma=s}^{\infty}$$

$$= \ln\left[\frac{\sigma+1}{\sigma+2}\right]_{\sigma=s}^{\infty}$$

$$= \ln\left[\frac{1+1/\sigma}{1+2/\sigma}\right]_{\sigma=s}^{\infty}$$

$$= \ln 1 - \ln\left[\frac{s+1}{s+2}\right] = \ln\left[\frac{s+2}{s+1}\right]$$

xvi)

$$L\left[\frac{\sinh 2t}{t}\right]$$

$$L[\sinh 2t]$$

$$= \frac{2}{s^2-4}$$

$$L\left[\frac{\sinh 2t}{t}\right]$$

$$= \int_{\sigma=s}^{\infty}\left(\frac{2}{\sigma^2-4}\right)$$

$$= 2\int_{\sigma=s}^{\infty}\left(\frac{1}{\sigma^2-4}\right)$$

$$= 2\tanh^{-1}\left(\frac{s}{4}\right)$$

$$= \tanh^{-1}\left(\frac{s}{2}\right)$$

i) $\dfrac{s-5}{(s-3)(s-4)} = \dfrac{A}{s-3} + \dfrac{B}{s-4}$

$s-5 = A(s-4) + B(s-3)$

when $s = 4$

$\cancel{4-5} = 4-5 = 0 + B$

$B = -1$

$s = 3$

$3-5 = -A + 0$

$A = 2$

$\dfrac{s-5}{(s-3)(s-4)} = \dfrac{2}{s-3} \mp \dfrac{1}{s-4}$

$L^{-1}\left[\dfrac{s-5}{(s-3)(s-4)}\right] = 2e^{3t} \mp e^{4t}$

ii) $\dfrac{2s-6}{(s-2)(s-4)} = \dfrac{A}{s-2} + \dfrac{B}{s-4}$

$2s-6 = A(s-4) + B(s-2)$

$s = 4$

$2(4)-6 = 0 + 2B$

$2 = 2B$

$B = 1$

$s = 2$

$2(2)-6 = A(2-4) + 0$

$-2 = -2A$

$A = 1$

$\dfrac{2s-6}{(s-2)(s-4)} = \dfrac{1}{s-2} + \dfrac{1}{s-4}$

$L^{-1}\left[\dfrac{2s-6}{(s-2)(s-4)}\right] = e^{2t} + e^{4t}$

iii) $\frac{5s-8}{s(s-4)} = \frac{A}{s} + \frac{B}{s-4}$

$5s-8 = A(s-4) + Bs$

$s = 0$

$-8 = -4A + 0$

$A = 2$

$s = 4$

$20 - 8 = 4B$

$12 = 4B$

$B = 3$

$\frac{5s-8}{s(s-4)} = \frac{3}{s} + \frac{3}{s-4}$

$L^{-1}\left[\frac{5s-8}{s(s-4)}\right] = 2 + 3e^{4t}$

iv) $\frac{s^2-3s-4}{(s-3)(s-1)^2} = \frac{A}{s-3} + \frac{B}{s-1} + \frac{C}{(s-1)^2}$

$s^2-3s-4 = A(s-1)^2 + B(s-3(s-1)) + C(s-3)$

$= A(s^2-2s+1) + B(s^2-s-3s+3) + C(s-3)$

$= A(s^2-2s+1) + B(s^2-4s+3) + C(s-3)$

$s^2 = As^2 + Bs^2$

$1 = A+B$

$-3s = -2As - 4Bs + Cs$

$-3 = -2A - 4B + C$

$-4 = A + 3B -$

$A = B+1$

$-3 = -2(B+1) - 4B + C$

$-3 = -2B - 2 - 4B + C$

$-3+2 = -6B + C$

$-1 = -6B + C$ — (2) ×3

$-3 = -18B + 3C$ — (3)

$-4 = B+1+3B$

$-5 = 4B - 3C$ —

(3) + (4)

$-8 = -14B = 0$

$B = \frac{8}{14} = \frac{4}{7}$

$$\frac{s}{s^2+a^2} \qquad \frac{s+2}{(s+2)^2+} \qquad \frac{s}{s^2+a^2}$$

v) $\dfrac{s-5}{s^2+4s+20}$

$$\frac{s-5}{s^2+4s+4+16} \quad \frac{s-5}{(s+2)^2+16}$$

$$= \frac{s+2-7}{(s+2)^2+16}$$

$$= \frac{s+2-7}{(s+2)^2+4^2}$$

$$= \frac{s+2}{(s+2)^2+4^2} - \frac{7}{(s+2)^2+4^2} = \frac{s+2}{(s+2)^2+4^2} - \left(\frac{4}{(s+2)^2+4^2}\right)\frac{7}{4}$$

$$L^{-1}\left[\frac{s+2}{(s+2)^2+4^2}\right] - L^{-1}\left[\left(\frac{4}{(s+2)^2+4^2}\right)\frac{7}{4}\right]$$

$$e^{-2t}\cos 4t - \frac{7}{4}e^{-2t}\sin 4t$$