Assignment 4.

① $(1-x^2)\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + 2y = 0$

$(1-x^2)y'' - 2xy' + 2y = 0$

Sub 1: $(1-x^2)y''$

$v = 1-x^2,\ v' = -2x,\ v'' = -2,\ v''' = 0$

$u^{(n)} = y^{(n+2)}$

$y^{(n)} = (1-x^2)y^{(n+2)} + (-2nx\, y^{(n+1)}) - 2\cdot\dfrac{n(n-1)}{2!}\, y^{(n)}$

Sub 2: $(-2x)y'$

$v = -2x,\ v' = -2,\ v'' = 0$

$u^{(n)} = y^{(n+1)}$

$y^{(n)} = -2xy^{(n+1)} - 2ny^{(n)}$

Sub 3: $2y$

$v = 2,\ v' = 0$

$u^{(n)} = y^{(n)}$

$y^{(n)} = 2y^{(n)}$

$\therefore\ y^{(n)} = (1-x^2)y^{(n+2)} - 2nxy^{(n+1)} - n(n-1)y^{(n)} - 2xy^{(n+1)} - 2ny^{(n)} + 2y$

$= (1-x^2)y^{(n+2)} - y^{(n+1)}(2nx + 2x) - y^{(n)}(n(n-1) + 2n - 2)$

at $x = 0$

$y^{(n)} = (y^{(n+2)}) - (y^{(n)})(n^2 + n - 2) = 0$

$\therefore\ (y^{n+2})_0 = (y^{n})_0(n^2+n-2)$

when $n = 0$: $(y^2)_0 = -2y_0$

$n = 1$: $(y^3)_0 = (0y')_0$

$n = 2$: $(y^4)_0 = (4y_0)^2 = 4(-2y_0)$

$n = 3$: $(y^5)_0 = 10y^3_0 = 10(0y')_0$

$n = 4$: $(y^{(6)})_0 = 18y^4_0 = (18)(4)(-2y_0)$

$y = y_0 + y'_0 x + y_0^2 \frac{x^2}{2!} + y_0^3 \frac{x^3}{3!} + y_0^4 \frac{x^4}{4!} + y_0^5 \frac{x^5}{5!} \cdots$

$= y_0 + y'_0 x + \left(-2y_0 \frac{x^2}{2!}\right) + (-2y_0)\frac{x^4}{4!} + (18 \times 4x - 2y_0)\frac{x^6}{6!} + \cdots$

$= y_0\left[1 - x^2 - \frac{x^4}{12} - \frac{x^6}{5}\right] + y'_0[x]$.

(2) $3e^{-4t} - 5e^{4t}$.

$\frac{3}{s+4} - \frac{5}{s-4}$

(ii) $\sin 4t + \cos 4t$

$\frac{4}{s^2+4^2} + \frac{s}{s^2+4^2} = \frac{4}{s^2+16} + \frac{s}{s^2+16}$

iii $t^2 + 2t^2 - t + 4$

$\frac{2}{s^3} + \frac{4}{s^3} - \frac{1}{s^2} + \frac{4}{s}$

iv $e^{-2t}\cos 5t$

$= \frac{s+2}{(s+2)^2+5^2} = \frac{s+2}{(s+2)^2+25}$

v. $t\sin 3t$

$= \frac{3}{s^2+3^2} = L\{\sin 3t\}$

$(-1)^n \frac{d^n}{ds^n} = -\frac{d}{ds}(f(s))$

$= -\frac{d}{ds}\left(\frac{3}{s^2+9}\right)$

$= \frac{(s^2+9)(0) - 3(2s)}{(s^2+9)^2} = \frac{-6s}{(s^2+9)^2} = \frac{-6s}{(s^2+9)^2}$

vi $\dfrac{e^{-t} - e^{-2t}}{t}$

$L\{e^{-t} - e^{-2t}\} = \dfrac{1}{s+1} - \dfrac{1}{s+2}$

$= \int_{\sigma = s}^{\infty} \left(\dfrac{1}{\sigma+1} - \dfrac{1}{\sigma+2} \right) d\sigma$

$= \left[\ln(\sigma+1) - \ln(\sigma+2) \right]_{\sigma=s}^{\infty}$

$= \ln\left(\dfrac{\sigma+1}{\sigma+2}\right)_{\sigma=s}^{\infty} = \ln\left(\dfrac{\infty+1}{\infty+2}\right) - \ln\left(\dfrac{s+1}{s+2}\right)$

$= \ln(1) - \ln\left(\dfrac{s+1}{s+2}\right)$

$= \ln\left(\dfrac{s+2}{s+1}\right)$

(vii) $e^{4t}\cos 2t$

$L\{e^{4t}\cos 2t\} = \dfrac{s-4}{(s-4)^2+2^2} = \dfrac{s-4}{(s-4)^2+4}$

viii $t\sin 2t$

$L\{\sin 2t\} = \dfrac{a}{s^2+a^2} = \dfrac{2}{s^2+4} = -\dfrac{d}{ds}\left(\dfrac{2}{s^2+4}\right)$

$= -\dfrac{(s^2+4)(0) - 2(2s)}{(s^2+4)^2} = \dfrac{-(-4s)}{(s^2+4)^2} = \dfrac{4s}{(s^2+4)^2}$

(ix) t^3+4t^2+5

$L\{t^3+4t^2+5\} = \dfrac{6}{s^4} + \dfrac{8}{s^3} + 5/s$

x $e^{3t}(t^2+4) \Rightarrow L\{f(t)\} = \dfrac{2}{s^3} + \dfrac{4}{s}$

$L\{e^{3t}(t^2+4)\} = \dfrac{2}{(s-3)^3} + \dfrac{4}{s-3}$

xi $t^2 \cos t$

$L\{\cos t\} = \frac{s}{s^2+1}$

$L\{t^2 \cos t\} = \frac{d}{ds}\left(-\frac{d}{ds}\left(\frac{s}{s^2+1}\right)\right)$

$= \frac{(s^2+1)(-4s)+2s^2(2s)}{(s^2+1)^2}$

$= \frac{-4s}{(s^2+1)^2}$

xii $\frac{\sinh 2t}{t} \Rightarrow \lim_{t \to 0} \frac{\cancel{t}\sinh 2 \times 0}{0} = \frac{0}{0}$ L'hopital's rule.

$L\{\sinh 2t\} = \frac{2}{s^2-4}$

$L\{\frac{m}{t}\} = \int_{\sigma=s}^{\infty} \frac{2}{\sigma^2-4}\, d\sigma$

$= 2\int_{\sigma=s}^{\infty} \frac{1}{\sigma^2-4}\, d\sigma$

$= 2\left(\frac{1}{2}\frac{\tan^{-1}\sigma}{2}\right)_{\sigma=s}^{\infty}$

$= \left(\frac{\tan^{-1}\sigma}{2}\right)_{\sigma=s}^{\infty}$

$= \left(\frac{\tan^{-1}\sigma}{2}\right)_{\sigma=s}^{\infty}$

$= \frac{\tan^{-1}\infty}{2} - \frac{\tan^{-1}s}{2} = \frac{-\tan^{-1}s}{2}$

3 (i) $\frac{s-5}{(s-3)(s-4)} = \frac{A}{(s-3)} + \frac{B}{(s-4)}$

$s-5 = A(s-4) + B(s-3)$

$s-5 = As - 4A + Bs - 3B$

$s-5 = As + Bs - 4A - 3B$

$A+B = 1 \quad --- (i)$

$-4A - 3B = -5 \quad -- (ii)$

$-4B - (-3B) = -4 - (-5)$

$-B = 1$

$\boxed{B = -1}$

$A - 1 = 1$

$\boxed{A = 2}$

$L^{-1}\left\{\frac{s-5}{(s-3)(s-4)}\right\} = L^{-1}\left\{\frac{2}{s-3} - \frac{1}{s-4}\right\}$.

$= 2e^{3t} - e^{4t}$

(ii) $\frac{2s-6}{(s-2)(-4)} = \frac{A}{s-2} + \frac{B}{s-4}$

$2s-6 = A(s-4) + B(s-2)$

$= As - 4A + Bs - 2B$

$A+B = 2 \quad --- (i)$

$-4A - 2B = -6 \quad --- (ii)$

$B = 1$

$A = 1$

$$L^{-1}\left\{\frac{7s-6}{(s-1)(s-4)}\right\} = L^{-1}\left\{\frac{1}{(s-1)} + \frac{1}{(s-4)}\right\}$$

$$= e^{t} + e^{4t}$$

(iii) $$\frac{5s-8}{s(s-4)} = \frac{A}{s} + \frac{B}{s-4}$$

$$5s-8 = A(s-4) + Bs$$
$$= As - 4A + Bs$$

$$-4A = -8$$
$$A = 2$$

$$5 = A + B$$
$$B = 5-2 = 3$$
$$\therefore L^{-1}\left\{\frac{5s-8}{s(s-4)}\right\} = 2 + 3e^{4t}$$

(iv) $$\frac{s^2-3s-4}{(s-3)(s-1)^2} = \frac{A}{s-3} + \frac{B}{s-1} + \frac{C}{(s-1)^2}$$

$$s^2-3s-4 = A(s-1)^2 + B(s-3)(s-1) + C(s-3)$$
$$= As^2 - 2As + A + Bs^2 - Bs - 3Bs + 3B + Cs - 3C$$

$$\left.\begin{aligned} A + B &= 1 \quad \cdots (1) \\ -2A - 4B + C &= -3 \quad \cdots (2) \\ A + 3B - 3C &= -4 \quad \cdots (3) \end{aligned}\right\} \begin{aligned} A &= -1 \\ B &= 2 \\ C &= 3 \end{aligned}$$

$$\frac{s^2-3s-4}{(s-3)(s-1)^2} = \frac{-1}{(s-3)} + \frac{2}{s-1} + \frac{3}{(s-1)^2}$$
$$L^{-1} = -e^{3t} + 2e^{t} + 3te^{t}$$

V $\frac{s-5}{s^2+4s+20} = \frac{s-5}{(s+2)^2+16}$

$= \frac{s}{(s+2)^2+16} - \frac{5}{(s+2)^2+16} = \frac{s+2-2}{(s+2)^2+4^2} - \frac{5}{(s+2)^2+4^2}$

$= \frac{s+2}{(s+2)^2+4^2} - \frac{7}{(s+2)^2+4^2} = \frac{s+2}{(s+2)^2+4^2} - \frac{7 \times 4/4}{(s+2)^2+4^2}$

$= e^{-2t}\left(\cos 4t - \frac{7}{4}\sin 4t\right)$