Assignment ~~Question~~ 4

1 $(1-x^2)\dfrac{d^2y}{dx} - 2x\dfrac{dy}{dx} + 2y = 0$

$(1-x^2)y^{(2)} - 2xy^{(1)} + 2y^{(0)} = 0$

~~W_1~~ = let $u^0 = y''$, $u' = y'''$, $u'' = y''''$

$v^0 = (1-x^2)$ $v' = -2x$ $v'' = -2$

$W_1 = y^{(n+2)}(1-x^2) + ny^{(n+1)}(-2x) + \dfrac{(-2)(n)(n-1)}{2!}y^n$

$W_2 = -\left(y^{(n+1)}(2x) + ny^{(n)}(2)\right)$

$y^{(n+2)}(1-x^2) \cancel{+} 2xny^{(n+1)} - n(n-1)y^n \cancel{+} 2xy^{(n+1)}$

$\cancel{+}2ny^{(n)} + 2y^{(n)} = 0$

$0 = y^{(n+2)}(1-x^2) + y^{(n+1)}(2xn \mp 2x) + y^{(n)}(-2n+2-n(n-1))$

when $x = 0$

$y^{(n+2)}(1-0) + y^{(n+1)}(0-0) + y^{(n)}(-n^2-n+2) = 0$

$y^{(n+2)} + y^{(n)}(-n^2-n+2) = 0$

$y^{(n+2)} = y^{(n)}(n^2+n-2) = 0$ - recurrent solution.

when $n=0$ $y^{(0+2)}_{(0)} = y^0_{(0)}(0+0-2) = y^0_{(0)}(-2) = -2y^0_{(0)}$

$n=1$ $y^{(3)}_{(0)} = y^1_{(0)}(1^2+1-2) = y'_{(0)}(0) = 0$

$n=2$ $y^{(4)}_{(0)} = y^2_{(0)}(2^2+2-2) = y^2_{(0)}(4) = 4\times -2y^0_{(0)}$

$n=3$ $y^{(5)}_{(0)} = y^3_{(0)}(3^2+3-2) = y^3_{(0)}(10) = 10\times 0$

$n=4$ $y^6_{(0)} = y^4_{(0)}(4^2+4-2) = y^4_{(0)}(18) = 18\times 4\times -2y^0$

$n=5$ $y^7_{(0)} = y^5_{(0)}(5^2+5-2) = y^5_{(0)}(28) = 28\times 10\times 0$

$n=6$ $y^8_{(0)} = y^6_{(0)}(6^2+6-2) = y^6_{(0)}(40) = 40\times 18\times 4\times -2y^0_{(0)}$

$y = y^0_{(0)} + xy'_{(0)} + \dfrac{x^2}{2!}.y^2_{(0)} + \dfrac{x^3}{3!}.y^3_{(0)} + \dfrac{x^4}{4!}y^4_{(0)} + \dfrac{x^5}{5!}y^5_{(0)}$

$+ \dfrac{x^6}{6!}y^6_{(0)} + \dfrac{x^7}{7!}y^7_{(0)} + \dfrac{x^8}{8!}y^8$

$$y = y^0_{(0)} + x y'_{(0)} + \frac{x^2}{2\times 1}(-2y^0) + \frac{x^4}{4\times 2\times 3\times 1}(4\times -2y^0) + \frac{x^6}{6\times 5\times 4\times 3\times 2\times 1}(18\times 4\times -2y^0)$$

$$\frac{x^8}{8\times 7\times 6\times 5\times 4\times 3\times 2\times 1}(40\times 18\times 4\times -2y^0)$$

$$y = y^0_{(0)}\left(1 - x^2 - \frac{x^4}{3} - \frac{x^6}{5} - \frac{x^8}{7} + \cdots +\right) + x y'_{(0)}$$

2) i $3e^{-4t} - 5e^{4t} = \dfrac{3}{s+4} - \dfrac{5}{s+4}$

ii $\sin 4t + \cos 4t = \dfrac{4}{s^2+16} + \dfrac{s}{s^2+16}$

iii $t^3 + 2t^2 - t + 4 = \dfrac{6}{s^4} + \dfrac{4}{s^3} - \dfrac{1}{s^2} + \dfrac{4}{s}$

iv $e^{-2t}\cos 5t = \dfrac{s}{(s+2)^2+25}$

v $t\sin 3t = -\dfrac{d}{ds}\left(\dfrac{3}{s^2+1}\right) = -\left[\dfrac{(s^2+1)(0) - 3)(2s)}{(s^2+1)^2}\right]$

$$= \frac{+6s}{(s^2+1)^2}$$

vi $\dfrac{e^{-t} - e^{-2t}}{t} = \displaystyle\int_6^\infty \frac{1}{6+1}\,d6 - \int_6^\infty \frac{1}{6+2}\,d6$

$$= \left[\ln(6+1) - \ln(6+2)\right]_s^\infty$$

$$= \left[\ln\left[\frac{6+1}{6+2}\right]\right]_s^\infty$$

$$= \left[-\ln\left[\frac{s+1}{s+2}\right] + \left[\ln\left[\frac{\infty}{\infty}\right]\right]\right]$$

$$= \left[\ln[s+2] - \ln[s+1]\right]$$

vii) $e^{4t}\cos 2t = \dfrac{s}{(s-4)^2+4}$

viii) $t\sin 2t = -\dfrac{d}{ds}\left[\dfrac{2}{(s^2+4)}\right] = -\left[\dfrac{(s^2+4)(0)-(2)(2s)}{(s^2+4)^2}\right]$

$= \dfrac{4s}{(s^2+4)^2}$

ix) $t^3+4t^2+5 = \dfrac{6}{s^4}+\dfrac{8}{s^3}+\dfrac{5}{s}$

xi $t^2\cos t = (-1)^2\dfrac{d^2}{ds^2}\left[\dfrac{s}{s^2+1}\right] = \dfrac{(s^2+1)(1)-(s)(2s)}{(s^2+1)^2}$

$= \dfrac{d}{ds}\left[\dfrac{-s^2+1}{(s^2+1)^2}\right] = \cancel{(s^2+1)^2(-2s)-(-s^2+1)}$

$t^2\cos t = \dfrac{(s^2+1)^2(-2s)-(-s^2+1)(4s^3+4s)}{(s^2+1)^4}$

xii $\dfrac{\sinh 2t}{t} = \int_6^\infty \dfrac{2}{6^2+4}\,d6$ $\quad u = 6^2+4$

$d6 = \dfrac{du}{26}$

$= \dfrac{1}{26}\int_6^\infty \dfrac{2}{u}\,d6 = \left[\dfrac{\ln u}{6}\right]_s^\infty$

$= \left[\dfrac{\ln(6^2+4)}{6}\right]_s^\infty$

$= \dfrac{\ln(\infty^2+4)}{6} - \dfrac{\ln(s^2+4)}{6}$

$=$

3i $\dfrac{s-5}{(s-3)(s-4)} = \dfrac{A}{s-3} + \dfrac{B}{s-4}$

$s = 3 \Rightarrow 3-5 = A(3-4) + 0$

$-2 = -A \;,\; A = 2$

$s = 4 \Rightarrow 4-5 = 0 + B(4-3)$

$-1 = B \;,\; B = -1$

$\dfrac{s-5}{(s-3)(s-4)} = \dfrac{2}{s-3} + \dfrac{1}{s-4} = 2e^{3t} - e^{4t}$

3ii) $\dfrac{2s-6}{(s-2)(s-4)} = \dfrac{A}{s-2} + \dfrac{B}{s-4}$

$s = 2 \Rightarrow 2(2)-6 = A(2-4) + 0$

$-2 = -2A \;,\; A = 1$

$s = 4 \Rightarrow 2(4)-6 = 0 + B(4-2)$

$2 = 2B \;,\; B = 1$

$\dfrac{2s-6}{(s-2)(s-4)} = \dfrac{1}{s-2} + \dfrac{1}{s-4} = e^{2t} + e^{4t}$

3iii) $\dfrac{5s-8}{s(s-4)} = \dfrac{A}{s} + \dfrac{B}{s-4}$

$s = 0 \Rightarrow 5(0)-8 = A(0-4) + 0$

$-8 = -4A \;,\; A = 2$

$s = 4 \Rightarrow 5(4)-8 = 0 + B(4)$

$12 = 4B \;,\; B = 3$

$\dfrac{5s-8}{s(s-4)} = \dfrac{2}{s} + \dfrac{3}{s-4} = 2 + 3e^{4t}$

3 iv) $\dfrac{s^2-3s-4}{(s-3)(s-1)^2} = \dfrac{A}{s-3} + \dfrac{B}{(s-1)^2} + \dfrac{C}{(s-1)}$

$s^2-3s-4 = A(s-1)^2 + B(s-3) + C(s-3)(s-1)$

$s^2-3s-4 = As^2-2As+A+Bs-3B+Cs^2-4Cs+C$

$s^2-3s-4 = s^2(A+C)+s(-2A+B-4C)+A-3B+C$

$A+C=1$

$-2A+B-4C=-3$

$A-3B+C=-4$

$A = -\dfrac{1}{3}$, $B = \dfrac{5}{3}$, $C = \dfrac{4}{3}$

$\dfrac{s^2-3s-4}{(s-3)(s-1)^2} = \dfrac{-1}{3(s-3)} + \dfrac{5}{3(s-1)^2} + \dfrac{4}{3(s-1)}$

$= \dfrac{1}{3}\left(-e^{3t} + 5te^{t} + 4e^{t}\right)$

3v) $\dfrac{s-5}{s^2+4s+20} = \dfrac{Ax+B}{s^2+4s+20}$

$s-5 = Ax+B(s^2+4s+20)$

$Ax+B=0$ $\qquad B=-Ax$

$4Ax+4B=1$ $\qquad 4Ax+4(-Ax)=1$

$20Ax+20B=-5$