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15/ENG07/046
PETROLEUM ENGINEERING

1. $(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$ Using Leibnitz Maclaurin Series

Solution

$(1-x^2)y'' - 2xy' + 2y = 0$

Sub 1

$(1-x^3)y''$

$u = y'' \therefore u^n = y^{n+2}$

$v = 1 - x^2$

$v' = -2x$

$v'' = -2$

$\therefore u^n v + n u^{n-1} v' + \frac{n(n-1)}{2!} u^{n-2} v^2 + \dots$

$y^n = y^{n+2}(1-x^2) + n y^{n+1} - 2x + \frac{n(n-1)}{2!} y^n - 2 + \dots$

$y^n = (1-x^2)y^{n+2} - 2nxy^{n+1} - n(n-1)y^n$

Sub 2

$-2xy'$

$u = y' \quad u^n = y^{n+1}$

$v = -2x, \quad v' = -2$

Using $u^n v + n u^{n-1} v' + \frac{n(n-1)}{2!} u^{n-2} v^2 + \dots$

$y^{n+1} \cdot -2x + n y^n \dots$

$= -2xy^{n+1} - 2ny^n$

Sub 3

$u = 2y$

$u = y \quad u^n = y^n$

$v = 2, \quad v' = 0$

$y^n =$ Using $u^n v + n u^{n-1} v' + \dots$

$y^n \cdot 2 = 2y^n$

Combining Sub 1, 2 & 3

$(1-x^2)y^{n+2} - 2nxy^{n+1} - n(n-1)y^n - 2xy^{n-1} - 2ny^n + 2y^n$

$(1-x^2)y^{n+2} + (-2nx - 2x)y^{n+1} + (-n^2 + n - 2n + 2)y^n = 0$

$(1-x^2)y^{n+2} - 2nx + 2xy^{n+1} - (n^2 - n + 2)y^n = 0$

$y^{(n+2)}$ at $x=0$

$-0\,(-n^2-n+2)y^n=0$

$y^{(n+2)} = (y^n)_0\,(n^2+n-2)$

$\therefore$ when $n=0$

$(y^2)_0 = (y^0)_0(-2) = -2(y^0)_0$

when $n=1$

$(y^3) = (y')_0\,(0) = 0$

when $n=2$

$(y^4)_0 = (y^2)_0(4) = 4(-2)(y^0)_0 = -8(y^0)_0$

when $n=3$

$(y^5) = (y^3)_0(10) = 10(y^2)_0 = 10\times 0 = 0$

when $n=4$

$(y^6)_0 = (y^4)_0(18) = 18(y^4)_0 = 18\times -8(y^0)_0 = -144(y^0)_0$

when $n=5$

$(y^7)_0 = (y^5)_0(28) = 28\times 0 = 0$

when $n=6$

$(y^8) = (y^6)_0(40) = 40\times -144(y^0)_0 = (-5760(y^0)_0$

$$\therefore y = y_0 + x(y')_0 + \frac{x^2}{2!}(y^2)_0 + \frac{x^3}{3!}(y^3)_0 + \frac{x^4}{4!}(y^4)_0 + \dots$$

$$y = y_0 + x(y')_0 + (-2y^0)_0 + \frac{x^2}{2!} + 0 + (-8y^0)\frac{x^4}{4!} + 0 + \frac{x^6}{6!}(-144)(y^0)_0$$

$$+ 0 + (-5760)(y^0)_0\frac{x^8}{8!}$$

$$y = y_0\left[1 - x^2 - \frac{x^4}{3} - \frac{x^6}{6} - \frac{x^8}{7}\right] + (y')_0(x)$$

i) $L\{3e^{-4t} - 5e^{4t}\} = \frac{3}{s+4} - \frac{5}{s-4}$

$= \frac{3(s-4) - 5(s+4)}{(s+4)(s-4)}$

$= \frac{3s - 12 - 5s - 20}{s^2 - 16} = \frac{-2s - 32}{s^2 - 16}$

ii) $L\{t^3 + 2t^2 - t + 4\} = \frac{3!}{s^4} + 2\left(\frac{2!}{s^3}\right) - \frac{1}{s^2} + \frac{4}{s}$

$= \frac{6}{s^4} + \frac{4}{s^3} - \frac{1}{s^2} + \frac{4}{s} = \frac{6 + 4s - s^2 + 4s}{s^4}$

4) $L\{e^{-2t} \cos 5t\}$

$L\{\cos 5t\} = \frac{5}{s^2 + 25}$

Let $s = s+2$

$L\{e^{-2t} \cos 5t\} = \frac{s+2}{(s+2)^2 + 25} = \frac{s+2}{s^2 + 4s + 29}$

i) $L\{t \sin 3t\}$

$L\{\sin 3t\} = \frac{3}{s^2 + 3^2} = \frac{3}{s^2 + 9}$

$-F'(s) = \frac{-d}{ds}\left[\frac{3}{s^2 + 9}\right]$

$\frac{s^2 + 9(0) - 3(2s)}{(s^2 + 9)^2} = \frac{-6s}{(s^2 + 9)^2}$

ii) $L\left\{\frac{e^{t} - e^{-2t}}{t}\right\}$

$\lim_{t \to 0} \frac{e^{-t} - e^{-2t}}{t} = \frac{e^{-0} - e^{-0}}{0} = \frac{1 - 1}{0} = 0$

Using L'hopital rule

$1. \{-e^{-t} + 2e^{-2t}\} = -1 + 2 \quad 1$

vii) $L\{e^{4t}\cos 2t\}$

$L\{\cos 2t\} = \frac{s}{s^2+2^2} = \frac{s}{s^2+4}$

let $s = s-4$

$L\{e^{4t}\cos 2t\} = \frac{s-4}{(s-4)^2+4} = \frac{s-4}{s^2-8s-12}$

viii) $L\{t\sin 2t\}$

$L\{\sin 2t\} = \frac{2}{s^2+2^2} = \frac{2}{s^2+4}$

$L\{t\sin 2t\} = -F'(s) = -\frac{d}{ds}\left[\frac{2}{s^2+4}\right]$

$\frac{s^2+4(0)-2(2s)}{(s^2+4)^2} = \frac{-4s}{(s^2+4)^2}$

ix) $L\{t^3+4t^2+5\}$

$= \frac{3!}{s^4} + \frac{4\cdot 2!}{s^3} + \frac{5}{s} = \frac{6+8s+5s^3}{s^4}$

x) $L\{e^{3t}(t^2+4)\}$

$L\{t^2e^{3t}\} + L\{4e^{3t}\}$

$L\{t^2\} = \frac{2!}{s^3}$

let $s = s-3$

$L\{e^{3t}t^2e^{3t}\} = \frac{2}{(s-3)^3}$

$L\{e^{3t}(t^2+4)\} = \frac{2}{(s-3)^3} + \frac{4}{s-3}$

$= \frac{2+4s^2-24s+36}{(s-3)^3} = \frac{4s^2-24s+38}{(s-3)^3}$

$L\{t^2 \cos t\}$

$L\{\cos t\} = \frac{s}{s^2+1^2} = \frac{s}{s^2+1}$

$-F'(s) = -\frac{d}{ds}\left[\frac{s}{s^2+1}\right]$

$\frac{[s^2+1][1] - s(2s)}{(s^2+1)^2} = \frac{s^2+1-2s^2}{(s^2+1)^2}$

$= \frac{-s^2+1}{(s^2+1)^2} = \frac{-1}{s^2+1}$

$L\{t^2 \cos t\} = -d/ds\left[\frac{-1}{s^2+1}\right]$

$\frac{(s^2+1)(0) - 1(2s)}{(s^2+1)^2} = \frac{-2s}{(s^2+1)^2}$

3) $\frac{s-5}{(s-3)(s-4)} = \frac{A}{(s-3} + \frac{B}{s-4}$

$A(s-4) + B(s-3) = s-5$

If $s=4$

$B(1) = -1$

$B = -1$

If $s=3$

$-A = -2$

$A = 2$

If $s=4$

$B(2) = 2(4) - 6 = 2$

$B = 1$

If $s = 2$

$A(2-4) = 2(2) - 6$

$-2A = -2$

$A = 1$

$\Rightarrow \frac{1}{s-2} + \frac{1}{s-4} = e^{2t} + e^{4t}$

(ii) $\frac{5s-8}{s(s-4)} = \frac{A}{s} + \frac{B}{s-4}$

$A(s-4) + B(s) = 5s - 8$

at $s = 0$

$-4A = -8$

$A = 2$

at $s = 4$

$4B = 12$

$B = 3$

$\Rightarrow \frac{2}{s} + \frac{3}{s-4} = 2 + 3e^{4t}$

(iv) $\frac{s^2 - 3s - 4}{(s-3)(s-1)^2} = \frac{A}{s-3} + \frac{B}{s-1} + \frac{C}{(s+1)^2}$

$A(s-1)^2 + B(s-3)(s-1) + C(s-3) = s^2 - 3$

at $s = 1$

$0 + 0 + C(-2) = 1^2 - 3(1) - 4$

$-2C = 6$

$C = 3$

at $s = 3$

$A + B = 1$

$B = 1 - A = 1 - (-1) = 1 + 1 = 2$

(2) $\Rightarrow -\frac{1}{s-3} + \frac{2}{s-1} + \frac{3}{(s-1)^2}$

$= e^{3t} + 2e^{t} + 3te^{t}$

4) $\frac{s-5}{s^2+4s-20} = \frac{s-5}{(s+2)^2+4^2}$

$= \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{16}$

$= A(s+2)(16) + B(16) + C(s+2)^2 = s-5$

$= (16s+32)A + 16B + C(s^2 + 4s + 4) = s-5$

Comparing Coefficients

$C = 0$

$16A + 4C = 1$

$32A + 16B + 4C = -5$

$\therefore C = 0$

$16A + 4C = 1$

$16A + 4(0) = 1$

$A = 1/16$

$32A + 16B + 4C = -5$

$32(1/16) + 16B + 4(0) = -5$

$2 + 16B = -5$

$16B = -5 - 2 = -7$

$B = \frac{-7}{16}$

$\Rightarrow \frac{1}{16(s+2)} - \frac{7}{16(s+2)^2} + 0$

$= \frac{1}{16} e^{-2t} - \frac{7}{16} te^{-2t}$