

ASSIGNMENT 4: Yusuf Yatakir Malgwi  
 15/ENG03/036  
 CIVIL ENG  
 300Level

①  $(1-x^2) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + 2y = 0$  — (1)

② can be written as  $(1-x^2)y'' - 2xy' + 2y = 0$  — (2)

taking the first term  $(1-x^2)y''$

$$u = y'' \quad v = 1-x^2$$

$$u' = y^{(n+2)} \quad v' = -2x$$

$$v'' = -2$$

$$v''' = 0$$

$$y^n = y^{(n+2)}(1-x^2) + n y^{(n+1)}(-2x) + n(n-1)y^{(n-2)}(-2) + n(n-1)(n-2)y^{(n-3)} \cdot 0$$

$$y^n = (1-x^2)y^{(n+2)} - 2xn y^{(n+1)} - 2n(n-1)y^{(n-2)} \quad (3)$$

taking the second term  $-2xy'$

$$u = y' \quad v = -2x$$

$$u' = y^{(n+1)} \quad v' = -2$$

$$v'' = 0$$

$$y^n = y^{(n+1)}(-2x) + n y^{(n)}(-2) + n(n-1)y^{(n-2)} \cdot 0$$

$$y^n = -2xy^{(n+1)} - 2ny^{(n)} + 0 \quad (4)$$

taking the third term  $2y$

$$u = y \quad v = 2$$

$$u' = y^n \quad v' = 0$$

$$y^n = y^n \cdot 2 + n y^{(n-1)} \cdot 0$$

$$y^n = 2y^n \quad (5)$$

∴ taking the three equations (3, 4 and 5) we have

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$$(1-x^2)y^{(n+2)} - 2xy^{(n+1)} - 2n(n-1)y^{(n)} - 2xy^{(n+1)} - 2ny^{(n)} + 2y^{(n)} = 0$$

$$(1-x^2)y^{(n+2)} - 2x(ny^{(n+1)} + y^{(n+1)}) + y^{(n)}(-2n + 2) = 0 \quad (6)$$

now at  $x=0$ , eqn 6 becomes

$$y^{(n+2)} + 2y^{(n)}(-n+1) = 0$$

$$y^{(n+2)} = -(-n+1)2y^{(n)}$$

$$y^{(n+2)} = (n-1)2y^{(n)} \quad (7)$$

eqn (7) gives the recurrence equation.

now at  $n=0$

$$y^{(2)} = (0-1)2y^{(0)}$$

$$(y^{(2)})_0 = -2y_0 \quad (8)$$

now at  $n=1$

$$(y^{(3)})_0 = 0 \quad (9)$$

now at  $n=2$

$$(y^{(4)})_0 = 2(y^{(2)})_0 = 2(-2y_0) = -4y_0 \quad (10)$$

now at  $n=3$

$$(y^{(5)})_0 = 4(y^{(3)})_0 = 4(0) = 0 \quad (11)$$

now at  $n=4$

$$(y^{(6)})_0 = 6y^{(4)} = 6(-4y_0) = -24y_0 \quad (12)$$

now at  $n=5$

$$(y^{(7)})_0 = 0 \quad (13)$$

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now at  $n=6$

$$(y^{(6)})_0 = 10(y^{(6)})_0 = 10(-2 + y_0) \quad \text{--- (13)}$$

$$= -2 + 0 y_0$$

Recall the Maclaurin method

$$y(x) = y_0(x) + xy'_0(x) + \frac{x^2}{2!} y''_0(x) + \frac{x^3}{3!} y'''_0(x) + \frac{x^4}{4!} y^{(4)}_0(x) + \frac{x^5}{5!} y^{(5)}_0(x) + \frac{x^6}{6!} y^{(6)}_0(x) + \dots = \frac{x^n}{n!} y^{(n)}_0(x)$$

now substitute eqn (7, 8, 9, 10, 11, 12 & 13) into eqn (4) we have?

$$y(x) = y_0(x) + xy'_0 + \frac{x^2}{2!} (-2y_0) + \frac{x^3}{3!} (0) + \frac{x^4}{4!} (-4y_0) + \frac{x^5}{5!} (0) + \frac{x^6}{6!} (-24y_0) + \frac{x^7}{7!} (0) + \frac{x^8}{8!} (-240y_0)$$

$$y(x) = y_0 \left( -2 \frac{x^2}{2!} - \frac{x^4}{4!} - \frac{24x^6}{6!} - \frac{240x^8}{8!} \right) + y_0 x + 0$$

$$y(x) = y_0 \left( -x^2 - \frac{x^4}{6} - \frac{x^6}{30} - \frac{x^8}{168} \right) + y_0 x$$

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② ①  $3e^{-4t} - 5e^{4t} = f(t)$   
- this is a linear equation so the Laplace transform

$$\begin{aligned} L\{f(t)\} &= f(s) \\ f(s) &= L\{3e^{-4t}\} - L\{5e^{4t}\} \\ f(s) &= 3 \frac{1}{s+4} - 5 \frac{1}{s-4} // \end{aligned}$$

② ②  $\sin 4t + \cos 4t = f(t)$

$$\begin{aligned} f(s) &= L\{f(t)\} \\ f(s) &= L\{\sin 4t\} + L\{\cos 4t\} \end{aligned}$$

$$f(s) = \frac{4}{s^2+16} + \frac{s}{s^2+16} //$$

② ③  $t^2 + 2t^2 - t + 4 = f(t)$

$$\begin{aligned} L\{f(t)\} &= f(s) \\ f(s) &= L\{t^2\} + 2L\{t^2\} - L\{t\} + L\{4\} \\ f(s) &= \frac{2!}{s^3} + 2 \left[ \frac{2!}{s^3} \right] - \frac{1}{s^2} + \frac{4}{s} \end{aligned}$$

$$f(s) = \frac{2}{s^3} + \frac{4}{s^3} - \frac{1}{s^2} + \frac{4}{s}$$

② ④  $e^{-2t} \cos 5t$

$$f(s) = L\{e^{-2t} \cos 5t\}$$

first taking the Laplace transform of  $\cos 5t$

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$$L(\cos 2t) = \frac{s}{s^2 + 2s} = \frac{s}{s(s+2)}$$

$$f(s) = \frac{s+2}{(s+2)^2 + 2s}$$

(v)  $t \sin 2t$   
 taking the Laplace transform of  $\sin 2t$   
 $= \frac{3}{s^2 + 9}$   
 now using the first derivative accordingly  
 to the power of  $t$  ( $n=1$ )  
 recall  $= -\frac{d}{ds} \frac{3}{s^2 + 9}$   
 using product rule  

$$-1 \left( \frac{s^2 + 9(0) - 3(2s)}{(s^2 + 9)^2} \right)$$

$$f(s) = -1 \left[ \frac{-6s}{(s^2 + 9)^2} \right]$$

$$f(s) = \frac{6s}{(s^2 + 9)^2} //$$

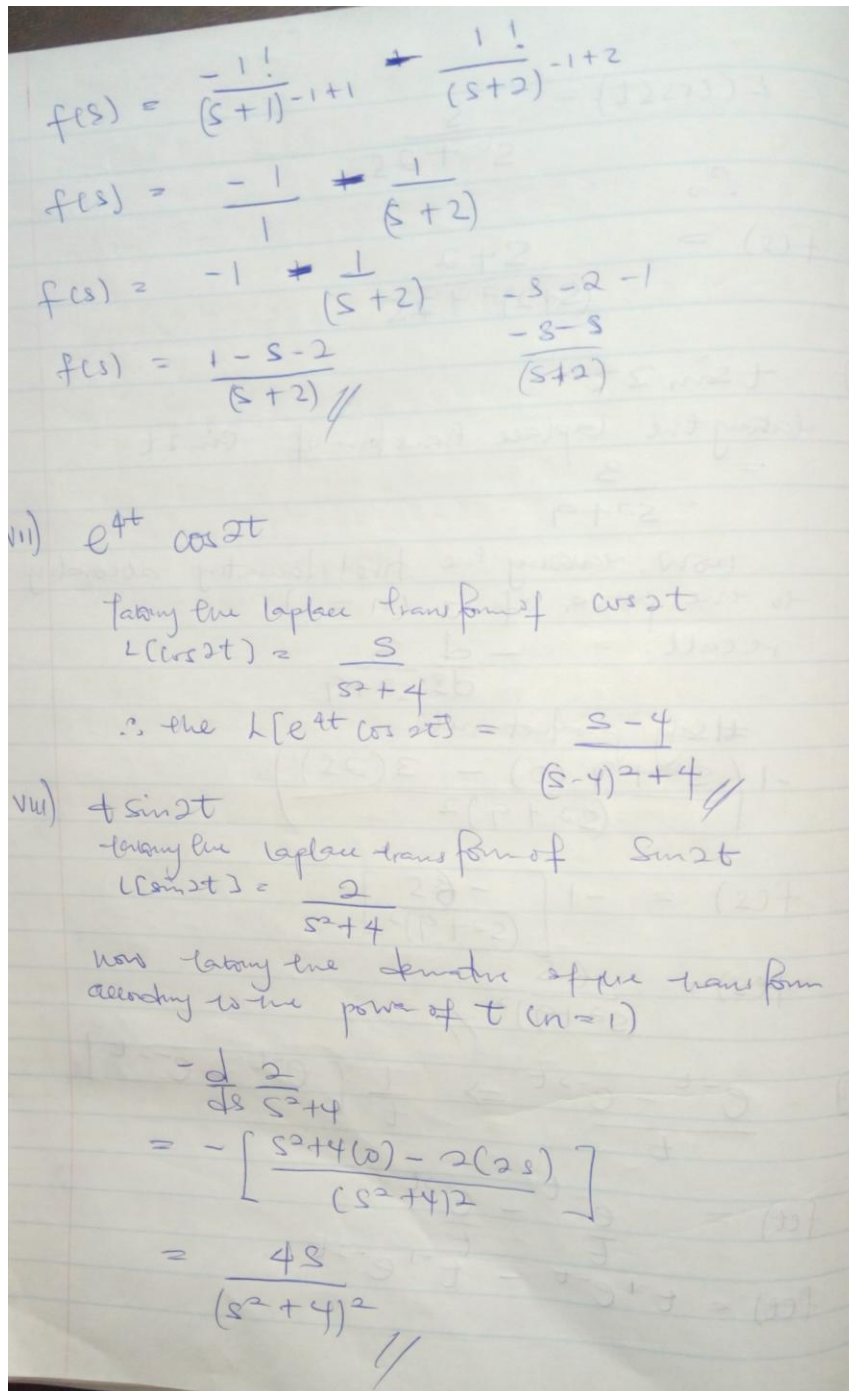
(vi)  $\frac{e^{-t} - e^{-2t}}{t} \Rightarrow \frac{1}{t} [e^{-t} - e^{-2t}]$   

$$f(t) = \frac{e^{-t}}{t} - \frac{e^{-2t}}{t}$$

$$f(s) = t^{-1} e^{-t} - t^{-1} e^{-2t}$$



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ex  $e^{at}(t^n)$   
 (1x)  $t^3 + 4t^2 + 5$   $f(t)$   
 $f(s) = L[f(t)]$   
 $f(s) = L[t^3] + L[4t^2] + L[5]$

$$f(s) = \frac{3!}{s^4} + \frac{4 \cdot 2!}{s^3} + \frac{5}{s}$$

$$f(s) = \frac{6}{s^4} + \frac{8}{s^3} + \frac{5}{s}$$

(x)  $t^2 \cos t$   
 taking the Laplace transform of  $\cos t$   
 $L[\cos t] = \frac{s}{s^2 + 1}$

now taking the derivative of the transform according to the power of  $t$  ( $n=2$ )

$$f(s) = -\frac{d}{ds} \left( -\frac{d}{ds} \frac{s}{s^2+1} \right)$$

$$= -\left[ \frac{(s^2+1)(1) - s(2s)}{(s^2+1)^2} \right]$$

$$= -\left[ \frac{s^2+1-4s}{(s^2+1)^2} \right]$$

$$= \frac{-1-s^2+4s}{(s^2+1)^2} = \frac{4s-s^2-1}{(s^2+1)^2}$$

now taking the second derivative

$$= -\left[ \frac{(s^2+1)^2(4-2s) - (4s-s^2-1)(4s^2+4s)}{(s^2+1)^4} \right]$$

$$= -\left[ \frac{(s^4+2s^2+1)(4-2s) - (4s-s^2-1)(4s^2+4s)}{(s^2+1)^4} \right]$$

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$$\begin{aligned}
 & - \left[ \frac{(s^4 + 2s^2 + 1)(4 - 2s) - (16s^3 + 16s^2 - 4s^5 - 4s^3 - 4s)}{(s^2 + 1)^2} \right] \\
 & - \left[ \frac{(s^4 + 2s^2 + 1)(4 - 2s) - (-4s^5 + 8s^3 + 16s^2 - 4s)}{(s^2 + 1)^2} \right] \\
 & - \left[ \frac{4s^4 + 2s^5 + 8s^2 - 4s^3 + 4 - 2s + 4s^5 - 8s^3 - 16s^2 + 4s}{(s^2 + 1)^2} \right] \\
 & - \left[ \frac{4s^4 + 2s^5 - 8s^2 - 12s^3 + 2s + 4}{(s^2 + 1)^2} \right] \\
 & = \frac{-4s^4 - 2s^5 + 8s^2 + 12s^3 - 2s - 4}{(s^2 + 1)^2}
 \end{aligned}$$



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NO 2

i)  $3e^{-4t} - 5e^{4t} = f(t)$   $f(s) = \frac{s+2}{(s+2)^2 + 25} //$

$L[f(t)] = f(s)$   
 $f(s) = L[3e^{-4t}] - L[5e^{4t}]$   
 $f(s) = \frac{3}{s+4} - \frac{5}{s-4} //$   
 $f(s) = \frac{3}{s+4} - \frac{5}{s-4} //$

ii)  $\sin 4t + \cos 4t = f(t)$   $f(s) = -1 \left[ \frac{s^2 + 9(0) - 2(0s)}{(s^2 + 9)^2} \right]$

$f(s) = L[\sin 4t] + L[\cos 4t]$   
 $f(s) = \frac{4}{s^2 + 16} + \frac{s}{s^2 + 16} //$   
 $f(s) = -1 \left[ \frac{-6s}{(s^2 + 9)^2} \right]$

iii)  $t^2 + 2t^2 - t + 4 = f(t)$   $f(s) = \frac{6s}{(s^2 + 9)^2} //$

$L[f(t)] = f(s)$   
 $f(s) = \frac{2!}{s^3} + 2 \left[ \frac{0!}{s^2} \right] - \frac{1}{s^2} + \frac{4}{s}$   
 $f(s) = \frac{2}{s^3} + \frac{4}{s^3} - \frac{1}{s^2} + \frac{4}{s} //$   
 $f(s) = \frac{-1!}{(s+1)^{-1+1}} + \frac{1!}{(s+2)^{-1+2}}$

iv)  $e^{-2t} \cos 5t = f(t)$   $f(s) = \frac{-1}{1} + \frac{1}{s+2}$

$f(s) = L[f(t)]$   
 $f(s) = -1 + \frac{1}{s+2}$   
 $f(s) = \frac{-s-3}{(s+2)} //$

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<p>vii) <math>e^{4t} \cos 2t = f(t)</math></p> $L[\cos 2t] = \frac{s}{s^2+4}$ $f(s) = \frac{s-4}{(s-4)^2+4} //$	<p>x) <math>e^{3t}(t^2+4) = f(t)</math></p> $\Rightarrow t^2 e^{3t} + 4e^{3t}$ $f(s) = -L(f'(t))$ $f(s) = \frac{2}{(s-3)^3} + \frac{4}{s-3} //$
<p>viii) <math>t \sin 2t</math></p> $L[\sin 2t] = \frac{2}{s^2+4}$ <p>Recall</p> $f(s) = -\frac{d^n}{ds^n} \frac{2}{s^2+4}$ $f(s) = -\left[ \frac{s^2+4(0) - 2(2s)}{(s^2+4)^2} \right]$ $f(s) = \frac{4s}{(s^2+4)^2} //$	<p>(xi) <math>t^2 \cos t</math></p> $f(s) = (-1)^n \frac{d^n}{ds^n}$ $L[\cos t] = \frac{s}{s^2+1}$ $f(s) = (-1)^2 \frac{d^2}{ds^2} \frac{s}{s^2+1}$ <p>taking the first derivative using quotient rule</p> $-1 \left[ \frac{s^2+1(1) - s(2s)}{(s^2+1)^2} \right]$ $\Rightarrow \frac{s^2+1-2s^2}{(s^2+1)^2} = \frac{-1+s^2}{(s^2+1)^2}$ <p>taking the second derivative</p> $\Rightarrow \frac{(s^2+1)^2(-2s) - (-1+s^2)(4s^2+4s)}{(s^2+1)^4}$
<p>ix) <math>t^3 + 4t^2 + 5 = f(t)</math></p> $f(s) = L(f(t))$ $f(s) = \frac{3!}{s^4} + 4 \frac{2!}{s^3} + \frac{5}{s}$ $+ \frac{5}{s}$ $f(s) = \frac{6}{s^4} + \frac{8}{s^3} + \frac{5}{s} //$	$\Rightarrow \frac{(3!)t^2 + (4!)t + 5}{(s^2+1)^4}$ $\frac{(s^2+1)^2(6s) + (s^2+1)(4s^2+4s)}{(s^2+1)^4}$ $\Rightarrow \frac{(s^2+1)[6s - 4s^2 + 4s]}{(s^2+1)^4}$

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$f(s) = \frac{6s - 4s^2}{(s^2 + 1)^2}$  //  
 (xii)  $\frac{\sinh at}{t} = f(s)$   
 $f(s) = t^{-1} \sinh at$   
 $L[\sinh at] = \frac{a}{s^2 - a^2}$   
 $f(s) = \frac{2 + 3}{s^2 + 4}$   
 $\frac{2 + 3s^2 + 12}{(s^2 + 4)(3s - 6)}$   
 $14 + 3s^2$   
 $f(s) = \frac{2s - 6}{(s-2)(s-4)}$   
 $\frac{A}{s-2} + \frac{B}{s-4} = \frac{2s-6}{(s-2)(s-4)}$   
 $A|_{s=2} = \frac{2(2)-6}{2-4} = 1$

(3)  
 i  $\frac{s-5}{(s-3)(s-4)} = f(s)$   
 $\frac{A}{s-4} + \frac{B}{s-3} = \frac{s-5}{(s-4)(s-3)}$   
 $A|_{s=4} = \frac{4-5}{4-3} = -1 = -1$   
 $B|_{s=3} = \frac{3-5}{3-4} = -\frac{2}{-1} = 2$   
 $A = -1$  and  $B = 2$   
 $f(s) = \frac{-1}{s-4} + \frac{2}{s-3}$   
 $f(t) = L^{-1}[f(s)]$   
 $f(t) = -e^{4t} + 2e^{3t}$   
 $f(t) = 2e^{3t} - e^{4t}$  //

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$B _{s=4} = \frac{2(4)-6}{(4-3)} = 1$ $A = 1 \text{ and } B = 1$ $f(s) = \frac{1}{s-2} + \frac{1}{s-4}$ $f(t) = \mathcal{L}^{-1}\{f(s)\}$ $f(t) = e^{2t} + e^{4t}$	$\text{iv) } \frac{s^3-3s-4}{(s-3)(s-1)^2} = f(s)$ $\frac{A}{s-3} + \frac{B}{s-1} + \frac{C}{s-1} = \frac{s^3+3s-4}{(s-3)(s-1)^2}$ $A _{s=3} = \frac{(3)^3-3(3)-4}{(3-1)^2}$ $= \frac{27-9-4}{4} = \frac{14}{4}$ $= \frac{7}{2}$ $B _{s=1} = \frac{d}{ds} \frac{s^3-3s-4}{s-3}$ $B _{s=1} = \frac{2s^3-9s^2+13}{(s-3)^2}$ $B _{s=1} = \frac{2-9+13}{4} = \frac{6}{4}$ $B = \frac{3}{2}$ $C _{s=1} = \frac{s^3-3s-4}{s-3}$ $= \frac{(1)^3-3(1)-4}{(1-3)} = 3$ $A = \frac{7}{2}, B = \frac{3}{2}, C = 3$ $f(s) = \frac{7}{2(s-3)} + \frac{3}{2(s-1)^2} + \frac{3}{s-1}$ $f(t) = \frac{7}{2}e^{3t} + \frac{3}{2}te^t + 3e^t$
$\text{iii) } \frac{5s-8}{s(s-4)} = f(s)$ $= \frac{A}{s} + \frac{B}{s-4} = \frac{5s-8}{s(s-4)}$ $A _{s=0} = \frac{5(0)-8}{(0-4)} = 2$ $B _{s=4} = \frac{5(4)-8}{4} = 3$ $A = 2 \text{ and } B = 3$ $f(s) = \frac{2}{s} + \frac{3}{s-4}$ $f(t) = \mathcal{L}^{-1}\{f(s)\}$ $f(t) = 2 + 3e^{4t}$	

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③ Convert the following function to time (t) domain!

(i)  $\frac{s-5}{(s-3)(s-4)}$

$$\frac{A}{(s-4)} + \frac{B}{(s-3)} = \frac{s-5}{(s-3)(s-4)}$$

$$A(s-3) + B(s-4) = s-5$$

$$A|s=4 = \frac{4-5}{(4-3)} = \frac{-1}{1} = -1$$

$$B|s=3 = \frac{3-5}{(3-4)} = \frac{-2}{-1} = 2$$

$$A = -1 \text{ and } B = 2$$

$$f(s) = \frac{-1}{(s-4)} + \frac{2}{(s-3)}$$

$$f(t) = \mathcal{L}^{-1}[f(s)]$$

$$f(t) = -1e^{4t} + 2e^{3t} //$$

(ii)

(ii)  $\frac{2s-6}{(s-2)(s-4)}$

$$\frac{A}{(s-2)} + \frac{B}{(s-4)} = \frac{2s-6}{(s-2)(s-4)}$$

$$A|s=2 = \frac{2(2)-6}{(2-4)} = \frac{-2}{-2} = 1$$

$$B|s=4 = \frac{2(4)-6}{(4-2)} = \frac{2}{2} = 1$$



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$A = 1$  and  $B = 1$   
 $f(s) = \frac{1}{(s-2)} + \frac{1}{(s-4)}$   
 $f(t) = \mathcal{L}^{-1}[f(s)]$   
 $f(t) = e^{2t} + e^{4t}$  //

iii  $\frac{5s-8}{s(s-4)}$   
 $\frac{A}{s} + \frac{B}{s-4} = \frac{5s-8}{s(s-4)}$   
 $A|_{s=0} = \frac{5(0)-8}{(0-4)} = \frac{-8}{-4} = 2$   
 $B|_{s=4} = \frac{5(4)-8}{4} = \frac{12}{4} = 3$   
 $A = 2$  and  $B = 3$   
 $f(s) = \frac{2}{s} + \frac{3}{s-4}$   
 $f(t) = \mathcal{L}^{-1}[f(s)]$   
 $f(t) = 2u(t) + 3e^{4t}$  //

(iv)  $\frac{s^3-3s-4}{(s-3)(s-1)^2}$   
 $\frac{A}{(s-3)} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)} = \frac{s^3-3s-4}{(s-3)(s-1)^2}$   
 $A|_{s=3} = \frac{(3)^3-3(3)-4}{(3-1)^2} = \frac{27-9-4}{4} = \frac{14}{4}$   
 $= \frac{7}{2}$

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$$B|_{s=1} = \frac{d}{ds} \frac{s^3 - 3s - 4}{(s-3)^2} \bigg|_{s=1} = \frac{(s-3)(3s^2 - 3) - (s^3 - 3s - 4)(2)(s-3)}{(s-3)^4} \bigg|_{s=1}$$

$$B|_{s=1} = \frac{3s^3 - 3s - 9s^2 + 9 - 2s^3 + 6s + 8}{(s-3)^2} \bigg|_{s=1}$$

$$B|_{s=1} = \frac{2s^3 - 9s^2 + 13}{(s-3)^2} \bigg|_{s=1} = \frac{2(1)^3 - 9(1)^2 + 13}{(1-3)^2} = \frac{2 - 9 + 13}{(-2)^2} = \frac{6}{4} = \frac{3}{2}$$

$$C|_{s=1} = \frac{s^3 - 3s - 4}{(s-3)} \bigg|_{s=1} = \frac{(1)^3 - 3(1) - 4}{(1-3)} = \frac{1 - 3 - 4}{-2} = \frac{-6}{-2} = 3$$

$$f(s) = \frac{7}{2(s-3)} + \frac{3}{2(s-1)^2} + \frac{3}{s-1}$$

$$f(t) = \frac{7}{2}e^{3t} + \frac{3}{2}e^t + 3e^t$$

(v)

$$\frac{s-5}{s^2+4s+20} = \frac{A}{(s+2-j4)} + \frac{B}{(s+2+j4)} = \frac{s-5}{(s+2-j4)(s+2+j4)}$$

$$A|_{s=-2+j4} = \frac{(-2+j4)-5}{(-2+j4+2+j4)} = \frac{-7+j4}{8j} = \frac{-7+j4}{8} \times \frac{j}{j}$$

$$A|_{s=-2+j4} = \frac{-7j-4}{-8} = \frac{7j+4}{8}$$

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$$B|_{s=-2-4j} = \frac{-2-4j-5}{(-2-4j+2-4j)} = \frac{-7-4j \times j}{-8j}$$
$$B|_{s=-2-4j} = \frac{-7j+4}{8}$$
$$f(s) = \frac{7j+4}{8(s+2-4j)} - \frac{7j+4}{8(s+2+4j)}$$
$$f(s) = \mathcal{L}^{-1}[f(s)]$$
$$f(t) = \frac{7j+4}{8} e^{(2+4j)t} - \frac{7j+4}{8} e^{(2-4j)t}$$
$$f(t) = \frac{7j+4}{8} (e^{(2+4j)t} - e^{(2-4j)t})$$