Assignment 4

1. $(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$

$\Rightarrow (1-x^2)y'' - 2xy' + 2y = 0$

let $(1-x^2)y'' = U_1$

$2xy' = U_2$

$2y = U_3$

for U_1 ; $V = 1-x^2$, $u = y^{(2)}$

$V^{(1)} = -2x$ $\quad u^{(n)} = y^{(n+2)}$

$V^{(2)} = -2$ $\quad u^{(n-1)} = y^{(n+1)}$

$V^{(3)} = 0$ $\quad u^{(n-2)} = y^{n}$

$U_1 = (1-x^2)y^{(n+2)} - n2xy^{(n+1)} - 2y^n \frac{n(n-1)}{2!}$

$= (1-x^2)y^{(n+2)} - n2xy^{(n+1)} - n(n-1)y^n$

for U_2 $\quad V = 2x$, $u = y^{(1)}$

$V^{(1)} = 2$ $\quad u^{(n)} = y^{(n+1)}$

$V^{(2)} = 0$ $\quad u^{(n-1)} = y^n$

$U_2 = 2xy^{(n+1)} + n2y^{(n)}$

for U_3 $\quad U_3 = y^{(n)}$

$(1-x^2)y^{(n+2)} - n2xy^{(n+1)} - n(n-1)y^n + 2xy^{(n+1)} + n2y^{(n)} + y^{(n)} = 0$

$(1-x^2)y^{(n+2)} - 2xy^{(n+1)}(n-1) - y^{(n)}(n(n-1) - 2n - 1) = 0$

$(1-x^2)y^{(n+2)} - 2xy^{(n+1)}(n-1) - y^{(n)}(n^2 - n - 2n - 1)$

$(1-x^2)y^{(n+2)} - 2xy^{(n+1)}(n-1) - y^{(n)}(n^2 - 3n - 1) = 0$

let $x = 0$

$y^{(n+2)} - (n^2 - 3n - 1)y^{(n)} = 0$

$y^{(n+2)} = (n^2 - 3n - 1)y^{(n)}$

$n = 0$; $y^{(2)} = 0$

$n = 1$; $y^{(3)} = y^{(1)}$

$n = 2$; $y^{(4)} = 0$

$n = 3$; $y^{(5)} = -3y^{3} = -3(y_0^1)$

$n = 4$; $y^{(6)} = 0$

$n = 5$; $y^{(7)} = -15y^5 = -15(-3y_0)$

$n = 6$; $y^{(8)} = 0$

$$y = y_0 + xy_0' + \frac{x^2}{2!}y_0'' + \frac{x^3}{3!}y_0''' + \frac{x^4}{4!}y_0^{iv} + \frac{x^5}{5!}y_0^{v} + \frac{x^6}{6!}y_0^{vi} + \frac{x^7}{7!}y_0^{vii} + \dots$$

$$y = y_0 + xy_0' + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(y_0'^3) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(-3y_0'^5) + 0 + \frac{x^7}{7!}[\dots] + 0$$

$$y = y_0 + xy_0' + \frac{x^3}{6}y_0'^3 - \frac{y_0'^5}{40}x^5 + \frac{13y_0'^7}{1008}x^7$$

$$\therefore y = y_0 + y_0'\left(x + \frac{x^3}{6} - \frac{x^5}{40} + \frac{x^7}{112} + \dots\right)$$

2 i. $L[3e^{-4t} - 5e^{4t}] = \frac{3}{s+4} - \frac{5}{s-4}$

ii. $L[\sin 4t + \cos 4t] = \frac{4}{s^2+16} + \frac{s}{s^2+16}$

iii. $L[t^3 + 2t^2 - t + 4] = \frac{3!}{s^{3+1}} + \frac{2(2!)}{s^{2+1}} - \frac{1!}{s^{1+1}} + \frac{4}{s}$

$$= \frac{6}{s^4} + \frac{4}{s^3} - \frac{1}{s^2} + \frac{4}{s}$$

iv. $L[e^{-2t}\cos 5t] = \frac{(s+2)}{(s+2)^2 + 25}$

v. $t \sin 3t$

$L[\sin 3t] = \frac{3}{s^2+9}$

$L[t \sin 3t] = -\frac{d}{ds}\left[\frac{3}{s^2+9}\right]$

vi. $\frac{e^{-t} - e^{-2t}}{t}$

$L[e^{-t} - e^{-2t}] = \frac{1}{s+1} - \frac{1}{s+2}$

$L\left[\frac{e^{-t}-e^{-2t}}{t}\right] = \int_{s}^{\infty} \frac{1}{s+1} - \frac{1}{s+2}\, ds$

$= \left[\ln(s+1) - \ln(s+2)\right]_s^{\infty}$

$= \left[\ln \frac{s+1}{s+2}\right]_s^{\infty}$

$= \ln\left[\frac{\infty+1}{\infty+2} - \frac{s+1}{s+2}\right]$

$= -\ln\left[\frac{s+1}{s+2}\right]$

$= \ln\left[\frac{s+2}{s+1}\right]$

vii. $L[e^{4t}\cos 2t] = \frac{(s-4)}{(s-4)^2+4}$

viii. $L[t\sin 2t] = -\frac{d}{ds}\left[\frac{2}{s^2+4}\right]$

ix. $L[t^3+4t^2+5] = \frac{3!}{s^{3+1}} + \frac{4(2!)}{s^{2+1}} + \frac{5}{s}$

$= \frac{6}{s^4} + \frac{8}{s^3} + \frac{5}{s}$

x. $e^{3t}(t^2+4)$

$L[t^2+4] = \frac{2!}{s^{2+1}} + \frac{4}{s}$

$= \frac{2}{s^3} + \frac{4}{s}$

$\therefore L[e^{3t}(t^2+4)] = \frac{2}{(s-3)^3} + \frac{4}{(s-3)}$

x) $t^2 \cos t$

$$L[\cos t] = \frac{s}{s^2+1}$$

$$\therefore L[t^2 \cos t] = \frac{d^2}{ds^2}\left(\frac{s}{s^2+1}\right)$$

xii) $\int_0^\infty \frac{\sinh 2t}{t}$

$u = s \quad v = s^2+1$

$du/ds = 1 \quad dv/ds = 2s$

$$= \frac{(s^2+1)1 - 2s(s)}{(s^2+1)^2}$$

$$= \frac{s^2+1-2s^2}{(s^2+1^2)^2}$$

$$= \frac{-s^2+1}{(s^2+1)^2}$$

$$-f''(s) = -\frac{d}{ds}\left(\frac{s^2-1}{(s^2+1)^2}\right)$$

$u = s^2-1 \quad v = (s^2+1)^2$

$du/ds = 2s \quad dv/ds = 4s(s^2+1)$

$$\frac{(s^2+1)^2 2s - (s^2-1)(4s^3+4s)}{(s^2+1)^4}$$

$$= \frac{-2s^5 - 4s^3 + 6s}{s^4+2s^2+1}$$

$$f''(s) = -\frac{d}{ds}\left(\frac{-2s^5-4s^3+6s}{s^4+2s^2+1}\right)$$

$$\therefore L\left[\frac{\sinh 2t}{t}\right] = \frac{2s^5+4s^3-6s}{s^4+2s^2+1}$$

3. $\dfrac{s-5}{(s-3)(s-4)} = \dfrac{A}{s-3} + \dfrac{B}{s-4}$

$= \dfrac{A(s-4) + B(s-3)}{(s-3)(s-4)}$

$s = 4$

$4 - 5 = A(4-4) + B(4-3)$

$-1 = B$

$B = -1$

$s = 3$

$3 - 5 = A(3-4) + B(3-3)$

$-2 = -A$

$A = 2$

$= \dfrac{2}{s-3} + \dfrac{-1}{s-4}$

$= \dfrac{2}{s-3} - \dfrac{1}{s-4}$

$= 2\left[\dfrac{1}{s-3}\right] - \left[\dfrac{1}{s-4}\right]$

$= 2e^{3t} - e^{4t}$

6. $\dfrac{2s-6}{(s-2)(s-4)} = \dfrac{A}{s-2} + \dfrac{B}{s-4}$

$2s - 6 = A(s-4) + B(s-2)$

$s = 4$

$2(4) - 6 = A(4-4) + B(4-2)$

$8 - 6 = 2B$

$2B = 2$

$B = 1$

$s = 2$

$2(2) - 6 = A(2-4) + B(2-2)$

$4 - 6 = -2A$

$-2 = -2A$

$A = 1$

$$= \frac{1}{s-2} + \frac{1}{s-6}$$

$$= e^{2t} + e^{6t}$$

iii $$\frac{5s-8}{s(s-4)} = \frac{A}{s} + \frac{B}{s-4}$$

$$5s-8 = A(s-4) + B(s)$$

$s = 4$

$$5(4) - 8 = A(4-4) + 4B$$

$$20 - 8 = 4B$$

$$4B = 12$$

$$B = 3$$

$s = 0$

$$5(0) - 8 = A(0-4)$$

$$-4A = -8$$

$$A = 2$$

$$\frac{2}{s} + \frac{3}{s-4}$$

$$= 2 + 3e^{4t}$$

iv $$\frac{s^2-3s-4}{(s-3)(s-1)^2} = \frac{A}{s-3} + \frac{B}{s-1} + \frac{C}{(s-1)^2}$$

$$s^2-3s-4 = A(s-1)^2 + B(s-3)(s-1) + C(s-3)$$

$$= A(s^2-2s+1) + B(s^2-4s+3) + C(s-3)$$

$$= \frac{-1}{s-3} + \frac{2}{s-1} + \frac{3}{(s-1)^2}$$

$$= -e^{-3t} + 2e^{t} + 3te^{t}$$

$$= -e^{-3t} + e^{t}(2+3t)$$

v) $\dfrac{s-5}{s^2+4s+20}$

$F(s) = \dfrac{s-5}{s^2+4s+20}$

$= \dfrac{s}{(s+2)^2+16} - \dfrac{5}{(s+2)^2+16}$

$= \dfrac{s+2-2}{(s+2)^2+16} - \dfrac{5}{(s+2)^2+16}$

$= \dfrac{s+2}{(s+2)^2+16} - \dfrac{2}{(s+2)^2+16}$

$= \dfrac{s+2}{(s+2)^2+4^2} - \dfrac{7 \times 4/4}{(s+2)^2+4^2}$

$= \dfrac{s+2}{(s+2)^2+4^2} - \dfrac{7}{4}\,\dfrac{4}{(s+2)^2+4^2}$

$= e^{-2t}\cos 4t - \frac{7}{4} e^{-2t} \sin 4t$

$= e^{-2t}\left[\cos 4t - \frac{7}{4} \sin 4t\right]$