i) $(1-x^2)\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + 2y = 0$

$(1-x^2)y'' - 2x\dfrac{dy'}{\cancel{d}} + 2y = 0$

$W_1 = (1-x^2)y''$

$u = y'' \qquad u^n = y^{(n+2)}$

$v = 1-x^2 \qquad v' = -2x \qquad v'' = -2 \qquad v''' = 0$

$W_1^n = y^{(n+2)}(1-x^2) + ny^{(n+1)}\cdot -2x + \dfrac{n(n-1)y^n\cdot -\cancel{2}}{\cancel{2}!} + 0$

$W_1^n = y^{(n+2)}(1-x^2) + -2xny^{(n+1)} - n(n-1)y^n$

$W_1^n = (1-x^2)y^{(n+2)} - 2xny^{(n+1)} - (n^2 - n)y^n$

$W_2 = -2xy'$

$v = -2x \qquad v' = -2 \qquad v'' = 0$

$u = y' \qquad u^n = y^{(n+1)}$

$W_2^n = y^{(n+1)}\cdot -2x + ny^n\cdot -2 + 0$

$W_2^n = -2xy^{(n+1)} - 2ny^n$

$W_3 = 2y$

$v = 2 \qquad v' = 0$

$u = y \qquad u^n = y^n$

$W_3^n = y^n\cdot 2 + 0$

$W_3^n = 2y^n$

$\therefore y^n = y^{(n+2)}(1-x^2) - 2xny^{(n+1)} - (n^2-n)y^n - 2xy^{(n+1)} - 2ny^n + 2y^n$

$y^n = (1-x^2)y^{(n+2)} - 2xny^{(n+1)} - 2xy^{(n+1)} - (n^2-n)y^n - 2ny^n + 2y^n$

$y^n = (1-x^2)y^{(n+2)} - 2xy^{(n+1)}(n+1) - y^n(n^2 - n + 2n - 2)$

$y^n = (1-x^2)y^{(n+2)} - (n+1)2xy^{(n+1)} - (n^2+n-2)y^n$

$\therefore$ at $x = 0$

$(1-0^2)y^{(n+2)} - \cancel{(n+1)2(0)y^{(n+1)}} - (n^2+n-2)y^n$

$y^{(n+2)} - (n^2+n+1)y^n$

$(y^{(n+2)})_0 = (n^2+n-2)y^n$

$\therefore$ at $n = 0$

$(y^2)_0 = (0^2+0-2)(y^0)_0$

$(y^2)_0 = -2(y^0)_0$

at $n = 1$

$(y^3)_0 = (1^2+1-2)(y')_0 = 0(y')_0 = 0$

at $n = 2$

$(y^4)_0 = (2^2+2-2)[y^2]_0 = 4[y^2]_0 = -2\cdot 4[y^0]_0 = -8[y^0]_0$

at $n = 3$

$[y^5] = (3^2+3-2)[y^3]_0 = 10[y^3]_0 = 13\cdot 3[y']_0 = 3[y]_0$

at $n = 4$

$[y^6] = (4^2+4-2)[y^4]_0 = 18[y^4]_0 = 18\cdot -8[y^0]_0 = 144[y^0]_0$

at $n = 5$

$[y^7] = (5^2+5-2)[y^5]_0 = 28[y^5]_0 = 28\cdot 3[y]_0 = 0\,120[y^0]_0$

$y = (y^0)_0 + x(y')_0 + \dfrac{x^2}{2!}(y^2)_0 + \dfrac{x^3}{3!}(y^3)_0 + \dfrac{x^4}{4!}(y^4)_0$

$+ \dfrac{x^5}{5!}(y^5)_0 + \dfrac{x^6}{6!}(y^6)_0 + \dfrac{x^7}{7!}(y^7)_0 + \cdots$

$$y = (y^0)_0 + x(y')_0 + \frac{x^2}{2}(y^0)_0 + \frac{x^3}{3\times 2}(y')_0 + \frac{x^4}{4\times 3\times 2}\cdot 7(y^0)_0$$

$$+ \frac{x^5}{5\times 4\times 3\times 2}\cdot 39(y')_0 + \frac{x^6}{6\times 5\times 4\times 3\times 2}\cdot 147(y^0)_0 + \frac{x^7}{7!}\cdot 1209(y')_0$$

$$y = (y^0)_0 + x(y')_0 + \frac{x^2}{2}(y^0)_0 + \frac{x^3}{2}(y')_0 + \frac{7x^4}{24}(y^0)_0 + \frac{13x^5}{40}(y')_0$$

$$+ \frac{49x^6}{240}(y^0)_0 + \frac{403x^7}{1608}(y')_0$$

$$y = (y^0)_0 + \frac{x^2}{2}(y^0)_0 + \frac{7x^4}{24}(y^0)_0 + \frac{49x^6}{240}(y^0)_0 + x(y')_0$$

$$+ \frac{x^3}{2}(y')_0 + \frac{13x^5}{40}(y')_0 + \frac{403x^7}{1608}(y')_0$$

$$y = (y^0)_0\left[1 + \frac{x^2}{2} + \frac{7x^4}{24} + \frac{49x^6}{240} + \cdots\right] + (y')_0\left[x + \frac{x^3}{2} + \frac{13x^5}{40} + \frac{403x^7}{1608} + \cdots\right]$$

$$y = (y^0)_0\left[1 - x^2 - \frac{x^4}{3} + \frac{3}{5}x^2\right] + x(y')_0$$

2) i) $L[3e^{-4t} - 5e^{4t}]$

$= L[3e^{-4t}] - L[5e^{4t}]$

$= \frac{3}{s+4} - \frac{5}{s-4}$

ii) $L[\sin 4t + \cos 4t]$

$= L[\sin 4t] + L[\cos 4t]$

$= \frac{4}{s^2+16} + \frac{s}{s^2+16}$

$= \frac{4+s}{s^2+16}$

$$\frac{2s-6}{(s-2)(s-4)} = \frac{A}{(s-2)} + \frac{B}{(s-4)}$$

$$\therefore 2s-6 = \cancel{7}\, A(s-4) + B(s-2)$$

at $s = 2$

$$2(2)-6 = A(2-4) + 0$$

$$4-6 = -2A$$

$$-2 = -2A$$

$$\therefore A = 1$$

at $s = 4$

$$2(4)-6 = \cancel{A(4}\, 0 + B(4-2)$$

$$8-6 = 2B$$

$$2 = 2B$$

$$\therefore B = 1$$

$$\therefore x(t) = L^{-1}\left[\frac{1}{(s-2)} + \frac{1}{(s-4)}\right]$$

$$x(t) = e^{2t} + e^{4t}$$

iii) $\dfrac{5s-8}{s(s-4)}$

$$x(t) = L^{-1}\left[\frac{A}{s} + \frac{B}{s-4}\right]$$

$$\therefore 5s-8 = A(s-4) + Bs$$

at $s = 0$

$$0-8 = A(0-4) + 0$$

$$\frac{-8}{-4} = \frac{-4}{-4}A$$

$$A = 2$$

at $s = 4$

$$5(4)-8 = 0 + 4B$$

$$20-8 = 4B$$

$$12 = 4B$$

$$B = 3$$

$\dfrac{B}{(s-4)}\Big]$

$$\therefore L^{-1}\left[\frac{2}{s} + \frac{3}{s-4}\right]$$

$$x(t) = 2 + 3e^{4t}$$

xii) $\frac{\sinh 2t}{t}$

$L[\sinh 2t] = \frac{2}{s^2-4}$

$x(s) = L\left[\frac{\sinh 2t}{t}\right] = \int_{\sigma=s}^{\infty} \frac{2}{s^2-4} = 2\int_{s}^{\infty} \frac{1}{\sigma^2-4}$

$= 2\left[\frac{1}{2}\tan^{-1}\frac{\sigma}{2}\right]_s^{\infty} = \left[\tan^{-1}\frac{\sigma}{2}\right]_s^{\infty}$

$= \tan^{-1}\frac{\infty}{2} - \tan^{-1}\frac{s}{2}$

$= \tan^{-1}\frac{2}{s}$

3] Convert the following to time (t) domains:

i) $\frac{s-5}{(s-3)(s-4)}$

$= L^{-1}\left[\frac{A}{(s-3)} + \frac{B}{(s-4)}\right]$

$\frac{s-5}{(s-3)(s-4)} = \frac{A}{(s-3)} + \frac{B}{(s-4)}$

$s-5 = A(s-4) + B(s-3)$

at $s=4$

$4-5 = 0 + B(4-3)$

$-1 = B \quad \therefore B = -1$

at $s=3$

$3-5 = A(3-4) + 0$

$-2 = -A$

$\therefore A = 2$

$x(t) = L^{-1}\left[\frac{2}{(s-3)} + \frac{-1}{(s-4)}\right]$

$\therefore x(t) = 2e^{3t} - e^{4t}$

ii) $\frac{2s-6}{(s-2)(s-4)} \quad \therefore x(t) = L^{-1}\left[\frac{A}{(s-2)} + \frac{B}{(s-4)}\right]$

$(s+4)^2 + 4$

viii) $t\sin 2t$

$L[\sin 2t] = \frac{2}{s^2+4}$

$L[t\sin 2t] = -1\frac{d}{ds}\left[\frac{2}{s^2+4}\right]$

$= -1 \times \frac{-4s}{(s^2+4)^2}$

$x(s) = \frac{4s}{(s^2+4)^2}$

ix) $t^3 + 4t^2 + 5$

$= L[t^3] + L[4t^2] + L[5]$

$= \frac{3!}{s^4} + \frac{4.2!}{s^3} + \frac{5}{s}$

$x(s) = \frac{6}{s^4} + \frac{8}{s^3} + \frac{5}{s}$

xi) $t^2\cos t$

$L[\cos t] = \frac{s}{s^2+1}$

$L[t^2\cos t] = (-1)^2\frac{d^2}{ds^2}\left[\frac{s}{s^2+1}\right]$

$= 1 \times \frac{2s^3-6s^3}{(s^2+1)^3}$

$= \cancel{\frac{2}{2}\left[\frac{s^3-3s}{2s^3+2s}\right]}$ $\quad \frac{2(s-3s^3)}{(s^2+1)^3}$

$x(s) = \cancel{\frac{s^3+3s}{2s^3+2s}}$ $\quad = \frac{2(s-3s^3)}{(s^2+1)^3}$

xv) $\cancel{t}\, e^{3t}(t^2+4)$

$L[t^2+4] = \frac{2}{s^3} + \frac{4}{s}$

$x(s) = L[e^{3t}(t^2+4)] = \frac{2}{(s-3)^3} + \frac{4}{s-3}$

iii) $L[t^3+2t^2-t+4]$

$= \frac{3!}{s^4} + 2\times\frac{2!}{s^3} - \frac{1}{s^2} + \frac{4}{s}$

$= \frac{6}{s^4} + \frac{4}{s^3} - \frac{1}{s^2} + \frac{4}{s}$

iv) $L[e^{-2t}\cos 5t]$

$\because L[\cos 5t] = \frac{s}{s^2+25}$

$L[e^{-2t}\cos 5t] = \frac{s+2}{(s+2)^2+25}$

v) $t\sin 3t$

$\because L[\sin 3t] = \frac{3}{s^2+9}$

$x(s) = L[t\sin 3t] = -1\frac{d}{ds}\left[\frac{3}{s^2+9}\right] = -1\times\frac{-6s}{(s^2+9)^2} = \frac{6s}{(s^2+9)^2}$

vi) $\frac{e^{-t}-e^{-2t}}{t}$

$L[e^{-t}-e^{-2t}] = \frac{1}{s+1} + \frac{1}{s+2}$

$L\left[\frac{e^{-t}-e^{-2t}}{t}\right] = \int_{\sigma=s}^{\infty}\frac{1}{s+1} - \frac{1}{s+2} = \int_{\sigma}^{\infty}\frac{1}{\sigma+1} - \frac{1}{\sigma+2}$

$= [\ln(\sigma+1) - \ln(\sigma+2)] = \ln\left[\frac{\sigma+1}{\sigma+2}\right]_{\sigma}^{\infty} = -\ln\frac{s+1}{s+2} = \ln\frac{s+2}{s+1}$

vii) $L[e^{4t}\cos 2t]$

$L[\cos 2t] = \frac{s}{s^2+4}$

$L[e^{4t}\cos 2t] = \frac{s+4}{(s+4)^2+4}$

iv) $\dfrac{s^2-3s-4}{(s-3)(s-1)^2}$

$$x(t) = L^{-1}\left[\frac{A}{(s-3)} + \frac{B}{(s-1)} + \frac{C}{(s-1)^2}\right]$$

$\cancel{x(t)}$ $s^2-3s-4 = A(s-1)^2 + B(s-3)(s-1) + C(s-3)$

at $s=3$

$3^2 - 3(3) - 4 = A(3-1)^2 + 0 + 0$

$9 - 9 - 4 = 4A$

$-4 = 4A$

$\therefore A = -1$

at $s = 1$

$1^2 - 3(1) - 4 = C(1-3)$

$-6 = -2C$

$C = 3$

$s^2 - 3s - 4 = As^2 - 2As + A + Cs - C3 + Bs^2 - B4s + B3$

Comparing Coefficients :

$A + 3B + 3C = -4$

$-1 + 3B - 3(3) = -4$

$3B = 10 - 4$; $3B = 6$ $\therefore B = 2$

$$\therefore x(t) = L^{-1}\left[\frac{-1}{(s-3)} + \frac{2}{(s-1)} + \frac{3}{(s-1)^2}\right]$$

$$x(t) = -e^{3t} + 2e^{t} + 3te^{t}$$

v) $\dfrac{s-5}{s^2+4s+20} = L^{-1}\Big[$