2.) $3e^{-4t} - 5e^{4t} = f(t)$

$L(f(t)) = F(s)$

$F(s) = L(3e^{-4t}) - L(5e^{4t})$

$= \frac{3}{s+4} - \frac{5}{s-4}$

$y = y^{o}_{(0)} + xy'_{(0)} + \frac{x^2}{2\times1}(-2y^o) + \frac{x^4}{4\times3\times2\times1}(4\times-2y^o) + \frac{x^6}{6\times5\times4\times3\times2\times1}(18\times4\times-2y^o) + \frac{x^8(40\times18\times4\times-2y^o)}{8\times7\times6\times5\times4\times3\times2\times1}$

$y = y^{o}_{(0)}\left(1 - x^2 - \frac{x^4}{3} - \frac{x^6}{5} - \frac{x^8}{7} + \cdots\right) + xy'(0)$

i.) $f(t) = 3e^{-4t} - 5e^{4t}$

$F(s) = \frac{3}{s+4} - \frac{5}{s-4}$

ii.) $\sin 4t + \cos 4t = f(t)$

$F(s) = \frac{4}{s^2+16} + \frac{s}{s^2+16}$

iii.) $f(t) = t^3 + 2t^2 - t + 4$

$F(s) = \frac{6}{s^4} + \frac{4}{s^3} - \frac{1}{s^2} + \frac{4}{s}$

iv.) $e^{-2t}\cos 5t = \frac{s}{(s+2)^2+25}$

v.) $t\sin 3t = -\frac{d}{ds}\left(\frac{3}{s^2+1}\right) = \frac{(s^2+1)(0) - (3)(2s)}{(s^2+1)^2}$

$= \frac{6s}{(s^2+1)^2}$

vii.)

viii.)

ix.)

vi) $f(t) = \frac{e^{-t} - e^{-2t}}{t}$

$$F(s) = \int_0^\infty \frac{1}{\sigma+1}\,d\sigma - \int_0^\infty \frac{1}{\sigma+2}\,d\sigma$$

$$= \Big(\ln(\sigma+1) - \ln(\sigma+2)\Big)_s^\infty$$

$$= \Big(\ln\Big(\frac{\sigma+1}{\sigma+2}\Big)\Big)_s^\infty$$

$$= \Big(-\ln\Big(\frac{s+1}{s+2}\Big) + \Big(\ln\Big(\frac{\infty}{\infty}\Big)\Big)\Big)$$

$$= \ln(s+2) - \ln(s+1)$$

G.S. = P.I. + C.F.

$y =$ P.I.

let $\sigma^2 + 4 = u$ $\quad \frac{du}{d\sigma} = 2\sigma$; $d\sigma = \frac{du}{2\sigma}$

$$\int_\sigma^\infty \frac{2}{u}\,\frac{du}{2\sigma} = \frac{2}{2\sigma}\ln u = \frac{1}{\sigma}\ln u$$

vii) $e^{4t}\cos 2t = \frac{s}{(s-4)^2+4}$

viii) $f(t) = t\sin 2t$

$$F(s) = -\frac{d}{ds}\left(\frac{2}{s^2+4}\right) = \frac{(s^2+4)(0) - (2)(2s)}{(s^2+4)^2}$$

$$= \frac{4s}{(s^2+4)^2}$$

ix) $f(t) = t^3 + 4t^2 + 5$

$$F(s) = \frac{6}{s^4} + \frac{8}{s^3} + \frac{5}{s}$$

x) $f(t) = e^{3t}(t^2+4) = t^2e^{3t} + 4e^{3t}$

$$F(s) = \frac{2}{(s-3)^3} + \frac{4}{s-3}$$

xi) $f(t) = t^2\cos t$

$$F(s) = (-1)^2\frac{d^2}{ds^2}\left(\frac{s}{s^2+1}\right) = \frac{(s^2+1)(1) - (s)(2s)}{(s^2+1)^2}$$

$$= \frac{d}{ds}\left(\frac{-s^2+1}{(s^2+1)^2}\right) = \frac{(s^2+1)^2(-2s) - (-s^2+1)(4s^3+4s)}{(s^2+1)^4}$$

xii) $\frac{\sinh 2t}{t}$

$$F(s) = \int_\sigma^\infty \frac{2}{\sigma^2+4}\,d\sigma$$ let $u = \sigma^2+4$ $\quad \frac{du}{d\sigma} = 2\sigma$ $\quad d\sigma = \frac{du}{2\sigma}$

$$= \frac{1}{2\sigma}\int_\sigma^\infty \frac{2}{u}\,d\sigma = \left(\frac{\ln u}{\sigma}\right)_s^\infty$$

$$= \left(\frac{\ln(\sigma^2+4)}{\sigma}\right)_s^\infty$$

$$= \frac{\ln(\infty^2+4)}{4} - \frac{\ln(s^2+4)}{\sigma}$$

3i) $\frac{s-5}{(s-3)(s-4)} = \frac{A}{s-3} + \frac{B}{s-4}$

$s = 3 \Rightarrow 3 - 5 = A(3-4) + 0$

$-2 = -A$; $A = 2$

If $s = 4$; $4 - 5 = B(4-3)$

$-1 = B$

$$\frac{s-5}{(s-3)(s-4)} = \frac{2}{s-3} - \frac{1}{s-4} = 2e^{3t} - e^{4t}$$

3ii.) $\frac{2s-6}{(s-2)(s-4)} = \frac{A}{s-2} + \frac{B}{s-4}$

if $s=2$; $2(2)-6 = A(2-4)$

$-2 = -2A$; $A=1$

if $s=4$; $2(4)-6 = B(4-2)$

$2 = 2B$; $B=1$

$\frac{2s-6}{(s-2)(s-4)} = \frac{1}{s-2} + \frac{1}{s-4} = e^{2t} + e^{4t}$

3iii.) $\frac{5s-8}{s(s-4)} = \frac{A}{s} + \frac{B}{s-4}$

if $s=0$; $5(0)-8 = A(0-4)$

$-8 = -4A$; $A=2$

if $s=4$; $5(4)-8 = B(4)$

$12 = 4B$; $B=3$

$\frac{5s-8}{s(s-4)} = \frac{2}{s} + \frac{3}{s-4} = 2 + 3e^{4t}$

3(iv) $f(t) = \frac{s^2-3s-4}{(s-3)(s-1)^2} = \frac{A}{s-3} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)^2}$

$s^2-3s-4 = A(s-1)^2 + B(s-3) + C(s-3)(s-1)$

$s^2-3s-4 = As^2 - 2As + A + Bs - 3B + Cs^2 - 4Cs + C$

$s^2-3s-4 = s^2(A+C) + s(-2A+B-4C) + A - 3B + C$

$A+C = 1$

$-2A+B-4C = -3$

$A-3B+C = -4$

$A = \frac{-1}{3}$, $B = \frac{5}{3}$, $C = \frac{4}{3}$

$\frac{s^2-3s-4}{(s-3)(s-1)^2} = \frac{-1}{3(s-3)} + \frac{5}{3(s-1)^2} + \frac{4}{3(s-1)}$

$= \frac{1}{3}(-e^{3t} + 5te^{t} + 4e^{t})$

v) $\frac{s-5}{s^2+4s+20} = \frac{Ax+B}{s^2+4s+20}$

$s-5 = Ax + B(s^2+4s+20)$

$Ax + B = 0$ $\qquad B = -Ax$

$4Ax + 4B = 1$ $\qquad 4Ax + 4(-Ax) = 1$

$20Ax + 20B = -5$

2.) $(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$

$(1-x^2)y^{(2)} - 2xy^{(1)} + 2y^{(0)} = 0$

let $u^0 = y''$, $u' = y'''$, $u'' = y''''$

$v^0 = (1-x^2)$, $v' = -2x$, $v'' = -2$

$W_1 = y^{(n+2)}(1-x^2) + ny^{(n+1)}(-2x) + \frac{(-2)(n)(n-1)y^n}{2!}$

$W_2 = -(y^{(n+1)}(2x) + ny^{(n)}(2))$

$y^{(n+2)}(1-x^2) - 2xny^{(n+1)} - n(n-1)y^n - 2x(y^{(n+1)} - 2ny^{(n)} + 2y^{(n)} = 0$

$0 = y^{(n+2)}(1-x^2) + y^{(n+1)}(2xn - 2x) + y^{(n)}(-2n+2-n(n-1))$

when $x=0$

$y^{(n+2)}(1-0) + y^{(n+1)}(0-0) + y^{(n)}(-n^2-n+2) = 0$

$y^{(n+2)} + y^{(n)}(-n^2-n+2) = 0$

$y^{(n+2)} = y^{(n)}(n^2+n-2) = 0$ recurrent solution

when $n=0$ $y^{(0+2)}_{(0)} = y^0_{(0)}(0+0-2) = y^0_{(0)}(-2) = -2y^0_{(0)}$

$n=1$ $y^{(3)}_{(0)} = y'_{(0)}(1^2+1-2) = y'_{(0)}(0) = 0$

$n=2$ $y^{(4)}_{(0)} = y^2_{(0)}(2^2+2-2) = y^2_{(0)}(4) = 4\times -2y^0_{(0)}$

$n=3$ $y^{(5)}_{(0)} = y^3_{(0)}(3^2+3-2) = y^3_{(0)}(10) = 10\times 0$

$n=4$ $y^6_{(0)} = y^4_{(0)}(4^2+4-2) = y^4_{(0)}(18) = 18\times 4\times -2y^0$

$n=5$ $y^7_{(0)} = y^5_{(0)}(5^2+5-2) = y^5_{(0)}(28) = 28\times 10\times 0$

$n=6$ $y^8_{(0)} = y^6_{(0)}(6^2+6-2) = y^6_{(0)}(40) = 40\times 18\times 4\times -2y^0_{(0)}$

$y = y^0_{(0)} + xy'_{(0)} + \frac{x^2}{2!}\cdot y^2_{(0)} + \frac{x^3}{3!}\cdot y^3_{(0)} + \frac{x^4}{4!}\cdot y^4_{(0)} + \frac{x^5}{5!}\cdot y^5_{(0)} + \frac{x^6}{6!}\cdot y^6_{(0)}$

$+ \frac{x^7}{7!}\cdot y^7_{(0)} + \frac{x^8}{8!}\cdot y^8_{(0)}$