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Q1 Use the Leibnitz-Maclaurin method to determine a series solution for

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$$

Soln

Let $w_1 = (1-x^2)\frac{d^2y}{dx^2}$

$v = 1-x^2, \quad v' = 2x, \quad v'' = -2, \quad v''' = 0$

$u = y'', \quad u' = y''', \quad u'' = y^{iv} \therefore u^n = y^{n+2}$

but $w_1 = u^n v + n u^{n-1} v' + \frac{n(n-1)}{2!} u^{n-2} v'' + \frac{n(n-1)(n-2)}{3!} u^{n-3} v'''$

$= y^{n+2} \cdot (1-x^2) + n \cdot y^{n+1} \cdot -2x + \frac{n(n-1)}{2!} \cdot y^n \cdot -2 + 0$

$\therefore w_1 = (1-x^2)y^{n+2} - 2xny^{n+1} - n(n-1)y^n$

Let $w_2 = -2x\frac{dy}{dx}$

$v = -2x, \quad v' = -2, \quad v'' = 0$

$u = y', \quad u' = y'' \therefore u^n = y^{n+1}$

but $w_2 = u^n v + n u^{n-1} v' + \frac{n(n-1)}{2!} u^{n-2} v''$

$= y^{n+1} \cdot -2x + n \cdot y^n \cdot -2 + 0$

$\therefore w_2 = -2xy^{n+1} - 2ny^n.$

Let $w_3 = 2y$,

$u^n = 2y^n$

but $w = w_1 + w_2 + w_3$

$\therefore w = (1-x^2)y^{n+2} - 2xny^{n+1} - n(n-1)y^n - 2xy^{n+1} - 2ny^n + 2y^n.$

At $x = 0$,

$y^{n+2} - n(n-1)y^n - 2ny^n + 2y^n = 0$

~~$y^{n+2} - n^2 + n$~~ $y^{n+2} + y^n(-n^2 - n + 2) = 0$

$\therefore y^{n+2} = -y_0^n(-n^2 - n + 2)$

At $n = 0, \quad y'' = -2 \cdot y_0$

$n = 1, \quad y''' = \cancel{-2 \cdot y_0'}\ 0$

$n = 2, \quad y^{iv} = -4 \cdot -y_0'' = 4 \cdot y_0'' = 4 \cdot -2 \cdot y_0$

$n = 3, \quad y^{v} = -10 \cdot -y_0''' = 10 \cdot y_0''' = 10 \cdot 0 = 0$

$n = 4, \quad y^{vi} = -18 \cdot -y_0^{iv} = 18 \cdot y_0^{iv} = 18 \cdot 4 \cdot -2 \cdot y_0$

$n = 5, \quad y^{vii} = -28 \cdot -y_0^{v} = 28 \cdot y_0^{v} = 28 \cdot 0$

From Maclaurin,

$$y(x) = y_0(x) + xy'_{(0)}(x) + \frac{x^2}{2!}y''_{(0)}(x) + \frac{x^3}{3!}y'''_{(0)}(x) + \frac{x^4}{4!}y^{iv}_0(x) + \frac{x^5}{5!}y^{v}_0(x)$$

$$= y_0 + xy'_0 + \frac{x^2}{\cancel{2}}\cdot -\cancel{2}^{-1}\cdot y_0 + \cancel{\frac{x^3}{3\cdot 2}\cdot 0}^{\,0} + \frac{x^4}{4\cdot 3\cdot \cancel{2}}\cdot \cancel{4}\cdot -\cancel{2}^{-1}\cdot y_0 + \cancel{\frac{x^5}{5\cdot 4\cdot 3\cdot 2}\cdot 0}^{\,0}$$

$$= y_0 + xy'_0 \cancel{+x} - x^2y_0 - \frac{x^4y_0}{3}$$

$$\Rightarrow y_0\left(1 - x^2 - \frac{x^4}{3} + \right) + y'_0(x + \cdots)$$

02 Transform the following into Laplace (s) domain:

i $3e^{-4t} - 5e^{4t}$

$$= \frac{3}{s+4} - \frac{5}{s-4} = \frac{3s - 12 - 5s - 20}{s^2 - 16}$$

$$= \frac{-2s - 32}{s^2 - 16}.$$

ii $\mathcal{L}\{\sin 4t + \cos 4t\}$

$$= \frac{4}{s^2 + 4^2} + \frac{s}{s^2 + 4^2} = \frac{4 + s}{s^2 + 16}.$$

iii $\mathcal{L}\{t^3 + 2t^2 - t + 4\}$

$$= \frac{3!}{s^4} + 2\cdot\frac{2!}{s^3} - \frac{1!}{s^2} + \frac{4}{s}$$

$$= \frac{6}{s^4} + \frac{4}{s^3} - \frac{1}{s^2} + \frac{4}{s} = \frac{6 + 4s - s^2 + 4s^3}{s^4}$$

$$\Rightarrow \frac{\cancel{-s^2}\, 4s^3 - s^2 + 4s + 6}{s^4}$$

iv $\mathcal{L}\{e^{-2t}\cos 5t\}$

$$= \frac{s+2}{(s+2)^2 + 25}.$$

v $\mathcal{L}\{t \sin 3t\}$ $= (-1)\dfrac{d}{ds}\left[\dfrac{3}{s^2+9}\right]$

$$= \cancel{\frac{9}{s^2}} \quad \cancel{\frac{3}{(s^2)+9}}$$

Let $u = 3,\ u' = 0$

$v = s^2 + 9,\ v' = 2s$

$$\text{but } \frac{d}{ds} = \frac{v\,du - u\,dv}{v^2} = \frac{\cancel{s^2+9}^{\,0}\cdot 0 - 3\cdot 2s}{(s^2+9)^2}$$

$$\therefore \mathcal{L}\{t \sin 3t\} = \frac{6s}{(s^2+9)^2}.$$

vi $\dfrac{e^{-t}-e^{-2t}}{t}$

$\lim\limits_{t\to\infty}\left[\dfrac{-e^{-t}+2e^{-2t}}{1}\right]$

$\cancel{L\{-e^{-t}+2e^{-2t}\}}$ $L\{-e^{-t}-e^{-2t}\} = \dfrac{1}{s+1} - \cancel{\dfrac{1}{s+2}}\ \dfrac{1}{s+2}$

vii $e^{4t}\cos 2t = \dfrac{s-4}{(s-4)^2+4}$.

viii $t\sin 2t = (-1)\dfrac{d}{ds}\left[\dfrac{4}{s^2+4}\right]$

$u = 4,\quad u' = 0$

$v = s^2+4,\quad v' = 2s$

$\dfrac{d}{ds} = \dfrac{v\cdot u' - u\cdot v'}{v^2} = \dfrac{\cancel{(s^2+4)}\,0 - 4\cdot 2s}{(s^2+4)^2}$

$\therefore L\{t\sin 2t\} = \dfrac{8s}{(s^2+4)^2}$

ix $L\{t^3+4t^2+5\} = \dfrac{3!}{s^4} + \dfrac{4\cdot 2!}{s^3} + \dfrac{5}{s}$

$= \dfrac{6}{s^4} + \dfrac{8}{s^3} + \dfrac{5}{s} = \dfrac{6+8s+5s^3}{s^4}$.

x $e^{3t}(t^2+4)$

$L\{t^2+4\} = \dfrac{2}{s^3} + \dfrac{4}{s}$.

$L\{e^{3t}\} = \dfrac{1}{s-3}$

$L\{e^{3t}(t^2+4)\} = \dfrac{2}{(s-3)^3} + \dfrac{4}{s-3} = \dfrac{2+4(s-3)^2}{(s-3)^3}$.

xi $L\{t^2\cos t\} = \dfrac{d^2 s}{d^2 s}\left[\dfrac{s}{s^2+1}\right]$

$u = s,\quad u' = 1$

$v = s^2+1,\quad v' = 2s$

$\dfrac{d}{ds} = \dfrac{s^2+1+2s^2}{(s^2+1)^2} = \dfrac{3s^2+1}{(s^2+1)^2}$

$\dfrac{d}{ds}\left(\dfrac{d}{ds}\right) \Rightarrow u = 3s^2+1,\quad u' = 6s;\quad v = (s^2+1)^2,\quad v' = 4s(s^2+1)$

$\frac{d^3}{ds^3} = \frac{(s^2+1)^3 \cdot 6s - (3s^2+1) \cdot 4s(s^2+1)}{[(s^2+1)^2]^3}$

xn $\frac{\sinh 2t}{t}$

$\lim_{t \to 0} \left[\frac{\sinh 2t}{t}\right] = \frac{2\cosh 2t}{1} = 2$

$\therefore L\left\{\frac{\sinh 2t}{t}\right\} = L\{\sinh 2t\}$

$= \frac{2}{s^2-2^2} \Rightarrow \frac{2}{s^2-4}$.

3 i. $\frac{s-5}{(s-3)(s-4)} = \frac{A}{s-3} + \frac{B}{s-4}$

$s-5 = A(s-4) + B(s-3)$

~~$s-5 = As-4A+$~~ Let $s=4$

~~$s-5 = B$ — (i)~~ $B=-1$

Let $s=3$,

$-A = -2$

$\therefore A = 2$.

$L^{-1}(x(s)) = \frac{2}{s-3} - \frac{1}{s-4}$

$\therefore x(t) = 2e^{3t} - e^{4t}$

ii. $\frac{2s-6}{(s-3)(s-4)} = \frac{A}{s-3} + \frac{B}{s-4}$

$2s-6 = A(s-4) + B(s-3)$

Let $s=4$,

$2 = 2B \quad \therefore B=1$

Let $s=2$,

$-2 = -2A \quad \therefore A=1$

$L^{-1}(x(s)) = \frac{1}{s-3} + \frac{1}{s-4}$

$\therefore x(t) = e^{3t} + e^{4t}$.

iii. $\frac{5s-8}{s(s-4)} = \frac{A}{s} + \frac{B}{s-4}$

$5s-8 = A(s-4) + Bs$

Let $A=4$,

$12 = 4B$

$\therefore B = 3$

$5s-8 = As - 4A + Bs$

$5s-8 = As - 4A + 3s$

$A+3 = 5$

$\therefore A = 2$.

$L^{-1}(x(s)) = \frac{2}{s} + \frac{3}{s-4}$

$\therefore x(t) = 2 + 3e^{4t}$

iv. $\frac{s^2-3s-4}{(s-3)(s-1)^2} = \frac{A}{s-3} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)}$

$s^2-3s-4 = A(s-1)^2 + B(s-3) + C(s-1)(s-3)$

Let $s=3$

$-4 = A(s^2-2s+1)$

$-4 \Rightarrow As^2 - 2As + A + Bs - 3B + Cs^2 - 4Cs + 3C$

$\Rightarrow (A+C)s^2 + s(-2A+3B-4C) + A - 3B + 3C$

$\because A+C = 1 \quad \therefore A = -C$

$-2A+3B-4C = -3$ — (i)

$A-3B+3C = -4$ — (ii)

from (i), $2C+B-4C = -3$

$-2C+B = -3$

$\therefore B = -3-2C$

from (ii), $-C+9+6C+3C = -4$

$9+8C = -4$

$8C = -13$

$C = -13/8$

$A = 13/8$

$B = -3 - \frac{26}{8} = \frac{-24-26}{8} = -\frac{50}{8}$

$\therefore B = -\frac{25}{4}$

$L^{-1}(x(s)) = \frac{13}{8}\left(\frac{1}{s-3}\right) - \frac{25}{4}\left(\frac{1}{(s-1)^2}\right) - \frac{13}{8}\left(\frac{1}{s-1}\right)$

$\therefore x(t) = \frac{13}{8}e^{3t} - \frac{25}{4}te^{t} - \frac{13}{8}e^{t}$

$$\frac{s-5}{s^2+4s+20} = \frac{s-5}{s^2+4s+4+16}$$

$$\Rightarrow \frac{s}{(s+2)^2+16} - \frac{5}{(s+2)^2+16}$$

$$= \frac{s+2-2}{(s+2)^2+16} - \frac{5}{(s+2)^2+16} = \frac{s+2}{(s+2)^2+16} - \frac{2}{(s+2)^2+16} - \frac{5}{(s+2)^2+16}$$

$$= \frac{s+2}{(s+2)^2+4^2} - \frac{7}{(s+2)^2+4^2}$$

$$= \frac{s+2}{(s+2)^2+4^2} - \frac{7}{4} \cdot \frac{4}{(s+2)^2+4^2}$$

$$\therefore x(t) = e^{-2t}\cos 4t - 7/4\, e^{-2t}\sin 4t.$$