

1 Mathematical Modelling

This is the process of setting up a model, solving it mathematically, and interpreting the result in physical or in order terms

ii using balance law - law of conservation of mass

b Forming a differential equation from an existing algebraic equation of the system.

$$r = (t^2 + 3t)i - 2 \sin 3t j + 3e^{2t} k$$

$$\frac{dr}{dt} = (2t + 3)i - 6 \cos 3t j + 6e^{2t} k$$

$$\frac{d^2 r}{dt^2} = \frac{d}{dt} \left(\frac{dr}{dt} \right)$$

$$\frac{d^2 r}{dt^2} = 2i + 18 \sin 3t j + 12e^{2t} k$$

$$\left. \frac{d^2 r}{dt^2} \right|_{t=0} = 2i + 18 \sin 3(0) j + 12e^{2(0)} k$$

$$= 2i + 18 \sin 0 j + 12e^0 k$$

$$= 2i + 18 \times 0 j + 12 \times 1 k$$

$$= 2i + 12k$$

$$\left| \frac{d^2 r}{dt^2} \right| = |2i + 12k|$$

$$= \sqrt{(2i)^2 + (12k)^2}$$

$$= \sqrt{4 \times i \cdot i + 144 \times k \cdot k}$$

$$= \sqrt{148} = 2\sqrt{37}$$

3

at point (1, 2, 1)

$$A = x^2 y i + (xy + yz) j + xz^2 k.$$

$$B = yz i - 3xz j + 2xy k.$$

$$\phi = 3x^2 y + yz^2 - 4y^2 z^2$$

$$\nabla \phi = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k$$

$$\nabla \phi = \frac{\partial}{\partial x} (3x^2 y + yz^2 - 4y^2 z^2 - 3) i + \frac{\partial}{\partial y} (3x^2 y + yz^2 - 4y^2 z^2 - 3) j +$$

$$\frac{\partial}{\partial z} (3x^2 y + yz^2 - 4y^2 z^2 - 3) k$$

$$\nabla \phi = (6xy + yz^2) i + (3x^2 + xz - 8yz^2) j + (xy - 8y^2 z) k.$$

at point (1, 2, 1)

$$\nabla \phi = (6(1)(2) + (1)(2^2)) i + (3(1)^2 + (1)(2) - 8(2)(1^2)) j + ((1)(2) - 8(2)^2(1)) k$$

$$\nabla \phi = 14i - 13j - 30k$$

$$\text{For } \nabla \cdot A = \frac{\partial}{\partial x} A_x + \frac{\partial}{\partial y} A_y + \frac{\partial}{\partial z} A_z$$

$$\nabla A = \frac{\partial}{\partial x} (x^2 y) + \frac{\partial}{\partial y} (xy + yz) + \frac{\partial}{\partial z} (xz^2)$$

$$= 2xy + (x + z) + 2xz$$

$$\nabla A (1, 2, 1) = 2(1)(2) + (1) + (1) + 2(1)(2)$$

$$4 + 2 + 2 = 8$$

$$\nabla A = 8$$

$$\nabla \times B = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & -3xz & 2xy \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y} \cdot 2xy - \frac{\partial}{\partial z} (-3xz) \right) i - \left(\frac{\partial}{\partial x} \cdot 2xy - \frac{\partial}{\partial z} \cdot yz \right) j + k$$

$$\left(\frac{\partial}{\partial y} \cdot 2xy - \frac{\partial}{\partial z} \cdot yz \right)$$

$$\vec{v} \times \vec{B} = (2x - (-3z))\vec{i} - \vec{j} (2y - 4) + \vec{k} (-3z - 2)$$

$$\vec{v} \times \vec{B} = 5x\vec{i} - y\vec{j} - \vec{k}(4z)$$

$$\vec{v} \times \vec{B} = 5x\vec{i} - y\vec{j} - 4z\vec{k}$$

$$1, 2, 1 = 5(1)\vec{i} - (2)\vec{j} - 4(1)\vec{k}$$

$$= 5\vec{i} - 2\vec{j} - 4\vec{k}$$

iv) $\text{grad} \cdot \text{div } A = \vec{\nabla} \cdot \vec{\nabla} A$

$$= \frac{\partial}{\partial x} \cdot \vec{\nabla} A \vec{i} + \frac{\partial}{\partial y} \cdot \vec{\nabla} A \vec{j} + \frac{\partial}{\partial z} \cdot \vec{\nabla} A \vec{k}$$

recall $\vec{\nabla} A = 2xy\vec{i} + x\vec{j} + z + 2xz\vec{k}$

$$\vec{\nabla} \cdot \vec{\nabla} A = \frac{\partial}{\partial x} (2xy + x + z + 2xz) \cdot \vec{i} + \frac{\partial}{\partial y} (2xy + x + z + 2xz) \cdot \vec{j} + \frac{\partial}{\partial z} (2xy + x + z + 2xz) \cdot \vec{k}$$

$$\vec{\nabla} \cdot \vec{\nabla} A = (2y + 1 + 2z) \cdot \vec{i} + (2x) \cdot \vec{j} + (1 + 2x) \cdot \vec{k}$$

at point (1, 2, 1)

$$= 2(2) + 1 + 2(1) \vec{i} + 2(1) \vec{j} + 1 + 2(1) \vec{k}$$

$$(4 + 1 + 2) \vec{i} + 2 \vec{j} + (1 + 2) \vec{k}$$

$$\vec{\nabla} \cdot \vec{\nabla} A = 7\vec{i} + 2\vec{j} + 3\vec{k}$$

$$\text{grad div } A = 7\vec{i} + 2\vec{j} + 3\vec{k}$$

$$\text{Curl}(\text{curl } A) = \text{curl}(\text{curl } A)$$

$$\text{curl } A = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & (xy + yz) & xz^2 \end{vmatrix}$$

$$\text{Curl } A = \vec{i} \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (xy + yz) & xz^2 \end{vmatrix} - \vec{j} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ xy & xz^2 \end{vmatrix} + \vec{k} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ xy & (xy + yz) \end{vmatrix}$$

Curl A

$$= \vec{i} \left(\frac{\partial}{\partial y} (xz^2) - \frac{\partial}{\partial z} (xy + yz) \right) - \vec{j} \left(\frac{\partial}{\partial x} (xz^2) - \frac{\partial}{\partial z} (xy) \right)$$

$$+ k \left(\frac{\partial}{\partial x} (xy + yz) - \frac{\partial}{\partial y} (x^2 y) \right)$$

$$= i(0 - y) + -j(z^2 - 0) + k(y - x^2)$$

$$\text{Curl } A = -yi - z^2 j + (y - x^2) k$$

$$\text{Curl } A = -yi - z^2 j + (y - x^2) k$$

$$\text{Curl Curl } A = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -yi & -z^2 & y - x^2 \end{vmatrix}$$

$$i \left(\frac{\partial}{\partial y} (y - x^2) - \frac{\partial}{\partial z} (-z^2) \right) - j \left(\frac{\partial}{\partial x} (y - x^2) - \frac{\partial}{\partial z} (-yi) \right)$$

$$+ k \left(\frac{\partial}{\partial x} (-z^2) - \frac{\partial}{\partial y} (-yi) \right)$$

Curl Curl A

$$= (i + 2z) i - j(-2zi - 0) + k(0 - (-1))$$

$$\text{Curl Curl } A = (1 + 2z) i + 2z j + k$$

$$\text{Curl Curl } A = (1 + 2z) i + 2z j + k$$

$$= (1 + 2(1)) i + 2(1) j + k$$

$$= 3 i + 2 j + k = 3 i + 2 j + k$$

$$\text{Curl Curl } A = 3 i + 2 j + k$$