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Mechanical
Eng 381

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = \cos t$$

Given that when $t = 0$, $x = 0{\cdot}1$

$$\frac{dx}{dt} = 0$$

(i) Using Auxiliary Method, obtain the solution in form of an expression having x as a function of t

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = \cos t \quad — ①$$

$x = CF + PI$

CF: $K^2 + 5K + 6 = 0$

$$K^2 + 2K + 3K + 6 = 0$$

$$K(K+2) + 3(K+2) = 0$$

$$(K+2)(K+3) = 0$$

$$K_1 = -2 \quad K_2 = -3$$

CF: $Ae^{k_1 t} + Be^{k_2 t}$

$$Ae^{-2t} + Be^{-3t}$$

PI: $x(t) = \cos t$

$$x = C\sin t + D\cos t$$

$$\frac{dx}{dt} = C\cos t - D\sin t$$

$$\frac{d^2x}{dt^2} = -C\sin t - D\cos t$$

$$-C\sin t - D\cos t + 5C\cos t - 5D\sin t + 6C\sin t + 6D\cos t = \cos t$$

$$(-C - 5D + 6C)\sin t + (-D + 5C + 6D)\cos t = \cos t$$

comparing coefficient

$$5C - 5D = 0 \quad —— ②$$

$$5C + 5D = 1 \quad — ③$$

from equation (2) $5C - 5D = 0$

(3) $5C + 5D = 1$

equation (3)−(2) $\Rightarrow$ $10D = 1$

$$D = \frac{1}{10}$$

Substitute $D = \frac{1}{10}$ into (2)

$$5C - 5\left(\frac{1}{10}\right) = 0$$

$$5C - \tfrac{1}{2} = 0$$

$$5C = \tfrac{1}{2}$$

$$C = \frac{1}{10}$$

PI $\quad C\sin t + D\cos t$

$$= \frac{1}{10}\sin t + \frac{1}{10}\cos t$$

$$x = CF + PI \Rightarrow Ae^{-2t} + Be^{-3t} + \frac{1}{10}\sin t + \frac{1}{10}\cos t$$

$$x = Ae^{-2t} + Be^{-3t} + \frac{1}{10}(\sin t + \cos t) \quad -(4)$$

$t = 0, \quad x = 0.1$

$$\frac{dx}{dt} = 0$$

$$\frac{dx}{dt} = -2Ae^{-2t} + (-3)Be^{-3t} + \frac{1}{10}(\cos t - \sin t) \quad (5)$$

from (4) $\quad x = 0.1 \quad t = 0$

$$0.1 = Ae^{-2(0)} + Be^{-3(0)} + \frac{1}{10}(\sin(0) + \cos(0))$$

$$A + B + \frac{1}{10} = 0.1$$

$$A + B = 0 \quad -(6)$$

from (5) $\quad \frac{dx}{dt} = 0, \; t = 0$

$$0 = -2Ae^{-2(0)} - 3Be^{-3(0)} + \frac{1}{10}(\cos(0) - \sin(0))$$

$$= -2A - 3B + \frac{1}{10}$$

$$2A + 3B = 0.1 \quad -(7)$$

Multiply eqn 6 by 2

$$2(A+B) = 0$$

$$2A + 2B = 0 \quad - 8$$

equation (7) - (8) :

$$\begin{array}{r} 2A + 3B = 0.1 \\ \underline{2A + 2B = 0} \\ B = 0.1 \\ B = 1/10 \end{array}$$

substitute $B = \frac{1}{10}$ into eqn (7)

$$2A + 2(1/10) = 1/10$$

$$2A + \frac{1}{5} = \frac{1}{10}$$

$$2A = \frac{1}{10} - 1/5 = \frac{-1}{10}$$

$$A = \frac{\frac{-1}{10}}{2} = -\frac{1}{20}$$

$$x = \frac{-1}{20}e^{-2t} + \frac{1}{10}e^{-3t} + \frac{1}{10}(\sin t + \cos t)$$

Write the steady state solution

$x = C\cos t + D\sin t$

let $C = k\cos a$ and

$D = k\sin a$

$\therefore x = (k\cos a)\cos t + (k\sin a)\sin t$

$x = k[\cos a(\cos t) + \sin a(\sin t)]$

applying trignometry

$x = k[\cancel{\sin t}\cos a + \cos t\sin a]$

$= k[\sin t\cos a - \cos t(-\sin a)$

$= k[\cos(\frac{\pi}{2} - t)\cos(-a) - \sin(\frac{\pi}{2} - t)\sin(-a)]$

$\sin(-t) = -\sin t \qquad \cos(-t) = \cos t$

$\sin t = \cos(\frac{\pi}{2} - t) \qquad \cos(t) = \sin(\frac{\pi}{2} - t)$

$x = k(\cos((\frac{\pi}{2} - t) + (-a))$

also $\cos(a+b) = \cos a\cos b - \sin a\sin b$.

$x = K[\cos((\frac{\pi}{2} - t) - a)]$

$= K\left[\cos\left(\frac{\pi}{2} - (t+a)\right)\right]$

$= k\left[\cos\frac{\pi}{2}\cos(t+a) + \sin\frac{\pi}{2}\sin(t+a)\right]$

also $\cos(a-b) = \cos a\cos b + \sin a\sin b$

$x = K[0 \times \cos(t+a) + 1 \times \sin(t+a)]$

$= K(\sin(t+a))$

$= K\sin(t+a)$

Therefore the steady state solution of $x = C\cos t + D\sin t$ is $x = K\sin(t+a)$