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ENG 381.
ASSIGNMENT 2.

1) If $y = e^{x^2+x}$

Show that

$$y'' = y'(2x+1) + 2y$$

and, hence, prove that

$$y^{(n+2)} = (2x+1)y^{(n+1)} + 2(n+1)y^n$$

Soln

To show that $y'' = y'(2x+1) + 2y$ —— (i)

Solve y' for $y = e^{x^2+x}$

$$y' = \frac{d}{dx}\left[e^{x^2+x}\right] = \frac{dy}{dx}$$

using Chain rule

let $u = x^2 + x$

$\therefore y = e^u$

$$\frac{dy}{dx} = \frac{du}{dx} \times \frac{dy}{du}$$

$$\frac{du}{dx} = 2x+1 \qquad \frac{dy}{dx} = e^u$$

$$\therefore \frac{dy}{dx} = e^u \cdot 2x+1 = e^{x^2+x}(2x+1)$$

$$\therefore y' = e^{x^2+x}(2x+1) \quad \text{—— (ii)}$$

To solve for y''

$$y'' = \frac{d}{dx}\left[e^{x^2+x} \cdot 2x+1\right] = \frac{d^2y}{dx^2}$$

Using the product rule

$$\frac{dy}{dx} = \frac{du}{dx} \cdot v + \frac{dv}{dx} \cdot u$$

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let $u = e^{x^2+x}$, $v = 2x+1$

$$\frac{du}{dx} = \frac{du}{da} \times \frac{da}{dx}$$

when

$a = x^2+x$

$$\frac{da}{dx} = 2x+1$$

$\therefore u = e^a$

$$\frac{du}{da} = e^a$$

$$\therefore \frac{du}{dx} = e^a \cdot 2x+1$$

Subs. $a = x^2+x$

$$\frac{du}{dx} = e^{x^2+x} \cdot 2x+1$$

$$\frac{dv}{dx} = 2$$

Since $y'' = \frac{d^2y}{dx^2}$

$$\therefore y'' = \frac{du}{dx} \cdot v + \frac{dv}{dx} \cdot u$$

$$y'' = e^{x^2+x} \cdot (2x+1) \cdot (2x+1) + 2 \cdot e^{x^2+x} \quad \text{— (iii)}$$

Sub. eqn (i) for y and eqn. (ii) for y' into eqn (iii)

$$y'' = y'(2x+1) + 2 \cdot y$$

$$\therefore y'' = y'(2x+1) + 2y$$

That is for $y' = e^{x^2+x} \cdot (2x+1)$ — (ii)

$y = e^{x^2+x}$ — (i)

Sub. (i) and (ii) into (iii)

$$y'' = y' \cdot (2x+1) + 2 \cdot y$$

Therefore, $y'' = y'(2x+1) + 2y$

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To prove $y^{(n+2)} = (2x+1)y^{(n+1)} + 2(n+1)y^{n}$

From

$$y'' = y'(2x+1) + 2y$$

$$y^{(2)} = y^{(1)}(2x+1) + 2y$$

Taking each term and differentiating n times using Leibnitz theorem

If $u_1 = y^{(2)}$

$u = y^{(2)}$ $\quad v = 1$

$u^{(n)} = y^{(n+2)}$ $\quad v^{(1)} = 0$

If $u_2 = y^{(1)}(2x+1)$

$u = y^{(1)}$ $\quad v = 2x+1$

$u^{(n)} = y^{(n+1)}$ $\quad v^{(1)} = 2$

$u^{(n-1)} = y^{(n)}$ $\quad v^{(2)} = 0$

If $u_3 = 2y$

$u = y$ $\quad v = 2$

$u^{(n)} = y^{(n)}$ $\quad v^{(1)} = 0$

From $y^{(n)} = u^{(n)}v + nu^{(n-1)}v^{(1)} + \frac{n(n-1)u^{(n-2)}v^{(2)}}{2!} + \cdots$

$u_1^{(n)} = y^{(n+2)} \cdot 1 + 0$

$u_2^{(n)} = y^{(n+1)} \cdot (2x+1) + n\,y^{(n)} \cdot 2$

$u_3^{(n)} = y^{(n)} \cdot 2 + 0$

Since $u_1^{(n)} = u_2 + u_3$

$\therefore u_1^{(n)} = u_2^{(n)} + u_3^{(n)}$

That is

$$y^{(n+2)} = y^{(n+1)}(2x+1) + 2n\,y^{(n)} + 2y^{(n)}$$

Collect like terms

$$y^{(n+2)} = (2x+1)y^{(n+1)} + (2n+2)y^{(n)}$$

$$y^{(n+2)} = (2x+1)y^{(n+1)} + 2(n+1)y^{(n)}$$

Therefore it has been proved that

$$y^{(n+2)} = (2x+1)y^{(n+1)} + 2(n+1)y^{(n)}$$

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2)
i) $y = x^3 e^{4x}$

To find $y^{(5)}$

From Leibnitz Theorem,

$$y^{(5)} = u^{(5)}v + 5u^{(5-1)}v^{(1)} + \frac{5(5-1)}{2!}u^{(5-2)}v^{(2)} + \frac{5(5-1)(5-2)u^{(5-3)}v^{(3)}}{3!} + \frac{5(5-1)(5-2)(5-3)u^{(1)}v^{(4)}}{4!}$$

$$+ uv^{(5)}$$

$$\therefore y^{(5)} = u^{(5)}v + 5u^{(4)}v^{(1)} + \frac{20 \cdot u^{(3)}v^{(2)}}{2!} + \frac{60u^{(2)}v^{(3)}}{3!} + \frac{120u^{(1)}v^{(4)}}{4!} + uv^{(5)}$$

$$y^{(5)} = u^{(5)}v + 5u^{(4)}v^{(1)} + 10u^{(3)}v^{(2)} + 10u^{(2)}v^{(3)} + 5u^{(1)}v^{(4)} + uv^{(5)}$$

when

$u = e^{4x}$ $\qquad$ $v = x^3$

$u^{(5)} = (4)^5 e^{4x}$ $\qquad$ $v^{(1)} = 3x^2$

$u^{(5)} = 1024 e^{4x}$ $\qquad$ $v^{(2)} = 6x$

$u^{(4)} = (4)^4 e^{4x} = 256 e^{4x}$ $\qquad$ $v^{(3)} = 6$

$u^{(3)} = (4)^3 e^{4x} = 64 e^{4x}$ $\qquad$ $v^{(4)} = 0$

$u^{(2)} = (4)^2 e^{4x} = 16 e^{4x}$ $\qquad$ $v^{(5)} = 0$

$u^{(1)} = (4)^1 e^{4x} = 4 e^{4x}$

$$y^{(5)} = 1024 e^{4x} \cdot x^3 + 5 \cdot 256 \cdot e^{4x} \cdot 3x^2 + 10 \cdot 64 e^{4x} \cdot 6x + 10 \cdot 16 e^{4x} \cdot 6 + 5 \cdot 16 e^{4x} \cdot 0 + e^{4x} \cdot 0$$

$$y^{(5)} = 1024 e^{4x} \cdot x^3 + 3840 e^{4x} x^2 + 3840 e^{4x} x + 960 e^{4x} + 0 + 0$$

The common factor on the R.H.S is $64 e^{4x}$

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$$y^{(5)} = 64e^{4x}\{16x^3 + 60x^2 + 60x + 15\}$$

ii) $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

This equation can be re-written as

$x^2 y'' + xy' + y = 0$

Taking each term and differentiating n times using Leibnitz theorem.

If $w_1 = x^2 y''$

let $u = y^{(2)}$ $\qquad v = x^2$

$u^{(n)} = y^{(n+2)}$ $\qquad v^{(1)} = 2x$

$u^{(n-1)} = y^{(n+1)}$ $\qquad v^{(2)} = 2$

$u^{(n-2)} = y^{(n)}$ $\qquad v^{(3)} = 0$

$\therefore w_1^{(n)} =$

From

$$y^{(n)} = u^{(n)}v + n u^{(n-1)}v^{(1)} + \frac{n(n-1)u^{(n-2)}v^{(2)}}{2!} + \cdots$$

$$w_1^{(n)} = y^{(n+2)} \cdot x^2 + n \cdot y^{(n+1)} \cdot 2x + \frac{n(n-1)y^{(n)} \cdot \cancel{2}}{\cancel{2}!} + 0$$

$$w_1^{(n)} = x^2 y^{(n+2)} + 2xn y^{(n+1)} + n(n-1)y^{(n)}$$

If $w_2 = xy'$

let $u = y^{(1)}$ $\qquad v = x$

$u^{(n)} = y^{(n+1)}$ $\qquad v^{(1)} = 1$

$u^{(n-1)} = y^{(n)}$ $\qquad v^{(2)} = 0$

$$w_2^{(n)} = y^{(n+1)} \cdot x + n y^{(n)} \cdot 1 + 0$$

If $w_3 = y$

$u = y$

$u^{(n)} = y^{(n)}$

$\therefore w_3^{(n)} = y^{(n)}$

Therefore $w_1^{(n)} + w_2^{(n)} + w_3^{(n)} = 0$

That is

$$x^2 y^{(n+2)} + 2xn y^{(n+1)} + n(n-1)y^n + y^{(n+1)}x + n y^{(n)} + y^{(n)} = 0$$

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$$x^2 y^{(n+2)} + x y^{(n+1)} (2n+1) + y^n (n^2 \cancel{-n} \cancel{+n} + 1) = 0$$

$$x^2 y^{(n+2)} + (2n+1) x y^{(n+1)} + (n^2+1) y^n = 0$$

This is what to show.

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