1) $y = e^{x^2+x}$, $y'' = y'(2x+1) + 2y$

let

$u = x^2 + x$, $y = e^u$

$\frac{\delta u}{\delta x} = 2x+1$, $\delta y/\delta u = ue^u$

$$\delta y/\delta u \times \frac{\delta u}{\delta x} = (2x+1)e^u$$

$$\delta y/\delta x = (2x+1)e^{(x^2+x)}$$

$$\therefore y' = (2x+1)(e^{x^2+x})$$

$$y''\ (\delta^2 y/\delta x^2) = (2x+1)[(2x+1)e^{x^2+x}] + 2(e^{x^2+x})$$

y'' where $\cancel{e^{x^2+}}$ $e^{x^2+x} = y$, $(2x+1)(e^{x^2+x}) = y'$

$$\therefore y'' = y'(2x+1) + 2(y)$$

(b) $y'' = y'(2x+1) + 2y$

$$y'' - y'(2x+1) - 2y = 0$$

from ~~us~~ leibnitz theorem

$W_1^n = y''$

$v_1 = 1$, $v_1' = 0$

$u^0 = y''$, $u^n = y^{n+2}$

$W_1^n = u^n v + n u^{n-1} v' + \frac{n(n-1)}{2!} u^{n-2} v'' + \cdots$

$W_1^n = y^{n+2}(1) + n y^{n+1}(0) + \cdots$

$\therefore W_1^n = y^{n+2}$

$-y'(2x+1) = -2xy' \mp y'$

$\cancel{W_2 = -y'}$. $\cancel{W_3 = 2x+1}$

$W_2 = -2xy'$ $W_3 = -y'$

$W_2^n = -2xy'$

$v^0 = -2x$, $v' = -2$, $v'' = 0$

i) $y = x^3 e^{4x}$, determine $y^{(5)}$

ii) $x^2 \frac{\delta^2 y}{\delta x^2} + x \frac{\delta y}{\delta x} + y = 0$, show that $x^2 y^{(n+2)} + (2n$
$+ (n^2+1) y^{(n)} = 0$

Sol

2ii) $x^2 y'' + x y' + y = 0$

Let;

$w_1 = x^2 y''$, $w_2 = x y'$, $w_3 = y$

$w_1 = x^2 y''$

$v = x^2$, $v' = 2x$, $v'' = 2$

$u = y^2$, $u' = 2y$, $u'' = y$

$\therefore u_n^{(n)} = y^{(n+2)}$

$w_1^{(n)} = \sum_{r=0}^{n} {}^nC_r \, u^{(n-r)} v^{(r)}$

$u^{(n)} v^{(0)} + n u^{(n-1)} v^{(1)} + \frac{n(n-1)}{2} u^{(n-2)} v^{(2)}$

$w_1^{(n)} = y^{(n+2)} x^2 + n y^{(n+1)} 2x + \frac{n(n-1)}{\cancel{2}} y^{(n)} \cancel{2}$

$w_1^{(n)} = x^2 y^{(n+2)} + 2nx y^{(n+1)} + n(n-1) y^{(n)}$

$w_2 = x y'$

$v = x$, $v' = 1$

$$y^{(n)} = u^n x^3 + n u^{n-1} 3x^2 + \frac{n(n-1)}{\cancel{2}} u^{n-2} \cdot \cancel{6}^{3} x + \frac{n(n-1)(n-2)}{\cancel{6}} u^{n-3} \cancel{6}$$

$$y^{(5)} = 1024e^{4x}x^3 + 5 \cdot 256 \cdot 3x^2 + 5(5-1)\,64e^{4x} + 5(5-1)(5-2)\,16e^{}$$

$$y^{(5)} = 1024e^{4x}x^3 + 3840x^2 + 1280e^{4x} + 960e^{4x}$$

$u = y^{(1)}, \; u^{(n)} = y''$

$u^{(n)} = y^{(n+1)}$

$x^2 y^{(n+2)} + (2n+1)x y^{(n+1)} + (n^2+1) y^{(n)} = 0$

$x^2 y'' + x y' + y = 0$

$w_1^{(n)} = u^{(n)} v + n u^{(n-1)} v^{(1)}$

$= y^{(n+1)} x + n y^{(n)} \cdot 1$

$w_2^{(n)} = x y^{(n+1)} + n y^{(n)}$

$w_3 = y$

$u = y, \; v = 1$

$u^{(n)} = y^{(n)}$

$w_3 = u^{(n)} v = y^{(n)} \cdot 1 = y^{(n)}$

$\therefore \quad w_1^{(n)} + w_2^{(n)} + w_3^{(n)} = 0$

$x^2 y^{(n+2)} + 2n x y^{(n+1)} + n(n-1) y^{(n)} + x y^{(n+1)} + n y^{(n)} + y^{(n)}$

$x^2 y^{(n+2)} + x(1+2n) y^{(n+1)} + (n(n-1) + n + 1) y^{(n)} = 0$

$x^2 y^{(n+2)} + x(2n+1) y^{(n+1)} + (n^2+1) y^{(n)} = 0$ //

2) $y = x^3 e^{4x}, \quad y^{(5)}$

$n = 5$

$v = x^3, \; u = e^{4x}$

$v^0 = x^3, \; v' = 3x^2, \; v^2 = 6x, \; v^3 = 6, \; v^4 = 0$

$u^0 = e^{4x}, \; u^{(1)} = 4e^{4x}, \; u^2 = 16e^{4x}, \; u^3 = 64e^{4x},$

$u^n = 4^n e^{4x} = u^5 = 4^5 e^{4x}$

$u = 1024 e^{4x}$

$u^0 = y'$, $u' = y''$, $u^n = y^{n+1}$

$W_2^n = u^n v + n u^{n-1} v' + \frac{n(n-1)}{2!} u^{n-2} v'' + \cdots$

$W_2^n = -2x y^{n+1} - 2n y^n$

$W_3^n = -y'$.

$u^0 = y'$, $u' = y''$, $u^n = y^{n+1}$

$v^0 = -1$, $v' = 0$

$W_3^n = y^{n+1}(-1) + 0 = -y^{n+1}$

$W_4^n = -2y$

$u^0 = y$, $u' = y'$, $u^n = y^n$

$v^0 = -2$, $v' = 0$

$W_4^n = -2y^n$

$u = y$, $u' = y'$, $u^n = y^n$

$v^0 = -2$, $v' = 0$

$\therefore$ $W_1^n + W_2^n + W_3^n + W_4^n$

$= y^{n+2} - 2x y^{n+1} - 2n y^n - (y^{n+1}) - 2y^n = 0$

$\therefore$ $y^{n+2} = (2x+1) y^{n+1} + 2(n+1) y^n$