$y = e^{x^2+x}$

Show ~~that~~ $y'' = y'(2x+1) + 2y$

$\cancel{\frac{dy}{dx} = y'}$

let $x^2+x = u$; $\frac{du}{dx} = (2x+1)$

$y = e^u \quad \frac{dy}{du} = e^u$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= e^u \times (2x+1)$$

$$= (2x+1)e^{x^2+x}$$

$$\frac{d^2y}{dx^2} = y'' = u\,dv/dx + v\,du/dx$$

$$= (2x+1)(2x+1)e^u + e^{x^2+x}(2)$$

Recall! : $y = e^{x^2+x}$

$y' = (2x+1)e^u$

$\therefore y'' = (2x+1)y' + 2y$

From y^{ll}

$$y^{(n)} = (2x+1)\, y^{(n-1)} + 2(n-1)\, y^{n}$$

$$i.e. \quad y^{(n+2)} = (2x+1)\, y^{(n+2-1)} + 2(n+2-1)\, y^{n}$$

$$y^{(n+2)} = (2x+1)\, y^{(n+1)} + 2(n+1)\, y^{n}$$

2) i) $y = x^3 e^{4x}$

$x^3 \to v$, $e^{4x} \to u$

$$\sum_{r=0}^{n} {}^nC_r\, u^{(n-r)} v^r$$

$v = x^3$, $v^{(1)} = 3x^2$, $v^{(2)} = 6x$, $v^{(3)} = 6$, $v^{(4)} = 0$, $v^{(5)} = 0$

$u = e^{4x}$, $u^{(1)} = 4e^{4x}$, $u^{(2)} = 16e^{4x}$, $u^{(3)} = 64e^{4x}$, $u^{(4)} = 256e^{4x}$, $u^{(5)} = 1024e^{4x}$

$$y^{(5)} = u^{(5)}v + 5u^{(5-1)}v^{(1)} + {}^5C_2 u^{(5-2)}v^{(2)} + {}^5C_3 u^{(5-3)}v^{(3)} + {}^5C_4 u^{(5-4)}v^{(4)}$$

$$= u^{(5)}v + 5u^{(4)}v^{(1)} + {}^5C_2 u^{(3)}v^{(2)} + {}^5C_3 u^{(2)}v^{(3)} + {}^5C_4 u^{(1)}v^{(4)}$$

$$= 1024e^{4x}x^3 + 5(256e^{4x})(3x^2) + 10(64e^{4x})6x + 10(16e^{4x})6 + 5(4e^{4x})(0)$$

$$= 1024e^{4x}x^3 + 3{,}840e^{4x}x^2 + 3{,}840e^{4x}x + 960e^{4x} + 0$$

$$y^{(5)} = e^{4x}(1024x^3 + 3840x^2 + 3840x + 960)$$

ii) $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = 0$; show $x^2y^{(n+2)} + (2n+1)xy^{(n+1)} + (n^2+1)y^{(n)} = 0$

$x^2y'' + xy' + y = 0$

$x^2y'' = \omega_1$

$v = x^2$, $v^{(1)} = 2x$, $v^{(2)} = 2$

$u = y''$, $u^{(1)} = y^{(3)}$, $u^{(2)} = y^{(4)}$ $\quad u^{n} = y^{(n+2)}$

$$\omega_1^{(n)} = \sum_{r=0}^{n} {}^nC_r\, u^{(n-r)} v^r$$

$$= u^{(n)}v + nu^{(n-1)}v^{(1)} + {}^nC_2 u^{(n-2)}v^{(2)} = u^{(n)}v + nu^{(n-1)}v^{(1)} + \frac{n(n-1)}{2}u^{(n-2)}v^{(2)}$$

$$= y^{(n+2)}x^2 + ny^{(n+1)}2x + \frac{n(n-1)y^{n}(2)}{2}$$

$\omega_1^{(n)} = y^{(n+2)} x^2 + n y^{(n+1)} 2x + n(n-1) y^n$

$x y' = \omega_2$

$v = x, \ v^{(1)} = 1$

$u = y^{(1)}, \ u^{(1)} = y^{(2)} \quad u^n = y^{(n+1)}$

$\omega_2 = u^{(n)} v + n u^{(n-1)} v^{(1)}$

$= y^{(n+1)} x + n y^{(n)}$

$\omega_3 = y'$

$v = 1, \ \cancel{v'} =$

$u = y' \quad u^{(n)} = y^{(n)}$

$\omega_3 = u^{(n)} v$

$\omega_3 = n y^n$

$\omega_1^{(n)} + \omega_2^{(n)} + \omega_3^{(n)} = 0$

$x^2 y^{(n+2)} + n y^{(n+1)} 2x + n(n-1) y^n + y^{(n+1)} x + n y^{(n)} + y^{(n)} = 0$

$x^2 y^{(n+2)} + x(1+2n) y^{(n+1)} + (n(n-1) + n + 1) y^{(n)} = 0$

$x^2 y^{(n+2)} + (2n+1) x y^{(n+1)} + (n^2 - n + n + 1) y^{(n)} = 0$

$x^2 y^{(n+2)} + (2n+1) x y^{(n+1)} + (n^2+1) y^{(n)} = 0$