1. If $y = e^{x^2+x}$, show that $y'' = y'(2x+1) + 2y$ and hence prove that $y^{(n+2)} = (2x+1)y^{(n+1)} + 2(n+1)y^{n}$

Answer

$y = e^{x^2+x}$

$\ln y = x^2 + x$

differentiating

$$\frac{1}{y}\frac{dy}{dx} = 2x+1$$

Multiplying by y

$$\frac{dy}{dx} = (2x+1)y$$

$$\frac{d^2y}{dx^2} = V\frac{du}{dx} + U\frac{dv}{dx}$$

$V = (2x+1)$ $\quad u = y$

$$\frac{dv}{dx} = 2 \qquad \frac{du}{dx} = \frac{du}{dy}\cdot\frac{dy}{dx}$$

$$\frac{du}{dy} = 1$$

$$\frac{du}{dx} = 1\cdot\frac{dy}{dx} = \frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = (2x+1)\cdot\frac{dy}{dx} + 2y$$

$$\frac{d^2y}{dx^2} = (2x+1)\frac{dy}{dx} + 2y$$

$y'' = (2x+1)y' + 2y$

$y'' - (2x+1)y' - 2y = 0$

$y^{(2)} - (2x+1)y^{(1)} - 2y = 0$

For $y^{(2)}$

nth derivative $= y^{(n+2)}$

For $(2x+1)y'$

$u = y'$ $\quad\quad$ $v = 2x+1$

$u^{n} = y^{(n+1)}$ $\quad\quad$ $v^{(1)} = 2$

$u^{(n-1)} = y^{n}$ $\quad\quad$ $v^{(2)} = 0$

Using leibnitz theorem

$y^{(n)} = u^{(n)}v + n\,u^{(n-1)}v^{(1)} + \dots$

For $2y$

nth derivative $= 2y^{n}$

$w_1^{(n)} = y^{(n+2)}$

$w_2^{(n)} = y^{(n+1)}(2x+1) + y^{n}2n$

$w_3^{(n)} = 2y^{n}$

$y^{(n+2)} - y^{(n+1)}(2x+1) - y^{n}2n - 2y^{n} = 0$

$y^{(n+2)} = y^{(n+1)}(2x+1) + 2ny^{n} + 2y^{n}$

$y^{(n+2)} = y^{(n+1)}(2x+1)\ 2y^{n}(n+1)$,,

(2) Using leibnitz theorem given that

(i) $y = x^{3}e^{4x}$, determine $y^{(5)}$

(ii) $x^{2}\dfrac{d^{2}y}{dx^{2}} + x\dfrac{dy}{dx} + y = 0$, show that $x^{2}y^{(n+2)} + (2n+1)xy^{(n+1)} + (n^{2}+1)y^{(n)} = 0$

Answer

(i) $y^{(n)} = u^{(n)}v + n\,u^{(n-1)}v^{(1)} + \dfrac{n(n-1)u^{(n-2)}v^{(2)}}{2!} + \dfrac{n(n-1)(n-2)u^{(n-3)}v^{(3)}}{3!}$

$u = e^{4x}$ $\quad\quad$ $v = x^{3}$

$u^{n} = 4^{5}e^{4x}$ $\quad\quad$ $v^{(1)} = 3x^{2}$

$u^{(n-1)} = 4^{(n-1)}e^{4x}$ $\quad\quad$ $v^{(2)} = 6x$

$u^{(n-2)} = 4^{(n-2)}e^{4x}$ $\quad\quad$ $v^{(3)} = 6$

$u^{(n-3)} = 4^{(n-3)}e^{4x}$ $\quad\quad$ $v^{(4)} = 0$

$u^{n} = 4^{5}e^{4x} = 1024e^{4x}$

$u^{(n-1)} = 4^{(5-1)}e^{4x} = 256e^{4x}$

$u^{(n-2)} = 4^{(5-2)}e^{4x} = 64e^{4x}$

$u^{(n-3)} = 4^{(5-3)}e^{4x} = 16e^{4x}$

$y^{(5)} = 1024e^{4x}x^{3} + 5\times256e^{4x}\times3x^{2} + \dfrac{5(5-1)\times64e^{4x}\times6x}{2!} + \dfrac{5(5-1)(5-2)\times16e^{4x}\times6}{3!}$

$y^{(5)} = 1024e^{4x}x^{3} + 3840e^{4x}x^{2} + \dfrac{7680e^{4x}x}{2!} + \dfrac{5760e^{4x}}{3!}$

$$y^{(5)} = 1024e^{4x}x^3 + 3840e^{4x}x^2 + 3840e^{4x}x + 960e^{4x}$$

$$y^{(5)} = e^{4x}(1024x^3 + 3840x^2 + 3840x + 960)$$

(ii) $x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = 0$

It can be written as

$$x^2y'' + xy' + y = 0$$

$$x^2 y^{(2)} + xy^{(1)} + y = 0$$

For $x^2 y^{(2)}$

$u = y^{(2)}$	$v = x^2$
$u^{(n)} = y^{(n+2)}$	$v^{(1)} = 2x$
$u^{(n-1)} = y^{(n+1)}$	$v^{(2)} = 2$
$u^{(n-2)} = y^{(n)}$	$v^{(3)} = 0$

For xy'

$u = y'$	$v = x$
$u^{(n)} = y^{(n+1)}$	$v^{(1)} = 1$
$u^{(n-1)} = y^{(n)}$	$v^{(2)} = 0$
$u^{(n-2)} = y^{(n-1)}$	

For y

nth derivative $= y^{(n)}$

Applying leibnitz theorem

$$y^{(n)} = u^{(n)}v + nu^{(n-1)}v^{(1)} + \frac{n(n-1)u^{(n-2)}v^{(2)}}{2!}$$

$$w_1^{(n)} = y^{(n+2)}x^2 + ny^{(n+1)}2x + \frac{n(n-1)y^{(n)}.2}{2!} + \frac{n(n-1)(n-2)y^{(n-1)}.0}{3!}$$

$$w_2^{(n)} = y^{(n+1)}.x + ny^{(n)}.1 + \frac{n(n-1)y^{(n-1)}.0}{2!}$$

$$w_3^{(n)} = y^{(n)}$$

$$w_1^{(n)} + w_2^{(n)} + w_3^{(n)} = 0$$

$$x^2y^{(n+2)} + 2nxy^{(n+1)} + n(n-1)y^{(n)} + xy^{(n+1)} + ny^{(n)} + y^{(n)} = 0$$

Collecting like terms

$$x^2y^{(n+2)} + xy^{(n+1)}(2n+1) + y^{(n)}(n^2 - n + n + 1) = 0$$

$$x^2y^{(n+2)} + xy^{(n+1)}(2n+1) + (n^2+1)y^{(n)} = 0$$