

$$1) \quad y = e^{2x^2+x}, \quad y'' = y'(2x+1) + 2y'$$

let
 $u = x^2 + x, \quad y = e^u$
 $\frac{\delta u}{\delta x} = 2x+1, \quad \frac{\delta y}{\delta u} = u e^u$

$$\frac{\delta y}{\delta u} \times \frac{\delta u}{\delta x} = (2x+1) e^u$$

$$\frac{\delta y}{\delta x} = (2x+1) e^{(x^2+x)}$$

$$\therefore y' = (2x+1) (e^{x^2+x})$$

$$y'' \left(\frac{\delta^2 y}{\delta x^2} \right) = (2x+1) \left[(2x+1) e^{x^2+x} \right] + 2(e^{x^2+x})$$

where $e^{x^2+x} = y, \quad (2x+1) e^{x^2+x} = y'$

$$\therefore y'' = y'(2x+1) + 2(y)$$

1b) $y'' = y'(2x+1) + 2y$
 $y'' - y'(2x+1) - 2y = 0$

from Leibnitz theorem

$$W_1^n = y''$$

$$v_1 = 1, \quad v_1' = 0$$

$$u^0 = y'', \quad u^n = y^{n+2}$$

$$W_1^n = u^n v + n u^{n-1} v' + \dots + \frac{n!}{2!} u^{n-2} v'' + \dots$$

$$W_1^n = y^{n+2} (1) + n y^{n+1} (0) + \dots$$

$$\therefore W_1^n = y^{n+2}$$

$$-y'(2x+1) = -2xy' - y'$$

$$W_2 = -y', \quad W_3 = 2x+1$$

$$W_2 = -2xy', \quad W_3 = -y'$$

$$W_2^n = -2xy'$$

$$v^0 = -2x, \quad v' = -2, \quad v'' = 0$$

$$u^0 = y', \quad u^1 = y'', \quad u^n = y^{n+1}$$

$$W_2^n = u^n v + n u^{n-1} v' + \frac{n(n-1)}{2!} u^{n-2} v'' + \dots$$

$$W_2^n = -2xy^{n+1} - 2ny^n$$

$$W_3^n = y'$$

$$u^0 = y', \quad u^1 = y'', \quad u^n = y^{n+1}$$

$$v^0 = -1, \quad v^1 = 0$$

$$W_3^n = y^{n+1} (n+1) + 0 = -y^{n+1}$$

$$W_4^n = -2y$$

$$u^0 = y, \quad u^1 = y', \quad u^n = y^n$$

$$v^0 = -2, \quad v^1 = 0$$

$$W_4^n = -2y^n$$

$$u = y, \quad u' = y', \quad u^n = y^n$$

$$v^0 = -2, \quad v^1 = 0$$

$$\therefore W_1^n + W_2^n + W_3^n + W_4^n$$

$$= y^{n+2} - 2xy^{n+1} - 2ny^n - (y^{n+1}) - 2y^n = 0$$

$$\therefore y^{n+2} = (2x(n+1))y^{n+1} + 2(n+1)y^n$$

i) $y = x^{\frac{1}{2}} e^{4x}$, determine $y^{(n)}$

ii) $x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = 0$, show that $x^2 y^{(n+2)} + (2n^2 + 1) y^{(n)} = 0$

Sol

2ii) $x^2 y'' + x y' + y = 0$

Let

$w_1 = x^2 y''$, $w_2 = x y'$, $w_3 = y$

$w_1 = x^2 y''$

$v = x^2$, $v' = 2x$, $v'' = 2$

$u = y^2$, $u' = 2y$, $u'' = y$

$\therefore u^{(n)} = y^{(n+2)}$

$w_1^{(n)} = \sum_{r=0}^n \binom{n}{r} u^{(n-r)} v^{(r)}$

$u^{(n)} v^{(0)} + n u^{(n-1)} v^{(1)} + \frac{n(n-1)}{2} u^{(n-2)} v^{(2)}$

$w_1^{(n)} = y^{(n+2)} x^2 + n y^{(n+1)} 2x + \frac{n(n-1)}{2} y^{(n)}$

$w_1^{(n)} = x^2 y^{(n+2)} + 2n x y^{(n+1)} + n(n-1) y^{(n)}$

$w_2 = x y'$

$v = x$, $v' = 1$

$$u = y^{(n)}, \quad u^{(n)} = y''$$

$$u^{(n)} = y^{(n+1)}$$

$$x^2 y^{(n+2)} + (2n+1)x y^{(n+1)} + (n^2+1)y^{(n)} = 0$$

$$x^2 y'' + x y' + y = 0$$

$$w_1^{(n)} = u^{(n)} \cdot v + n u^{(n-1)} \cdot v^{(1)} \\ = y^{(n+1)} x + n y^{(n)} \cdot 1$$

$$w_2^{(n)} = x y^{(n+1)} + n y^{(n)}$$

$$w_3 = y$$

$$u = y, \quad v = 1$$

$$u^{(n)} = y^{(n)}$$

$$w_3 = u^{(n)} \cdot v = y^{(n)} \cdot 1 = y^{(n)}$$

$$\therefore w_1^{(n)} + w_2^{(n)} + w_3^{(n)} = 0$$

$$x^2 y^{(n+2)} + 2n x y^{(n+1)} + n(n-1) y^{(n)} + x y^{(n+1)} + n y^{(n)} + y^{(n)} \\ x^2 y^{(n+2)} + x(1+2n) y^{(n+1)} + (n(n-1) + n + 1) y^{(n)} = 0$$

$$x^2 y^{(n+2)} + x(2n+1) y^{(n+1)} + (n^2+1) y^{(n)} = 0 //$$

$$2i) \quad y = x^3 e^{4x}, \quad y^{(5)}$$

$$n = 5$$

$$v = x^3, \quad u = e^{4x}$$

$$v^0 = x^3, \quad v^1 = 3x^2, \quad v^2 = 6x, \quad v^3 = 6, \quad v^4 = 0$$

$$u^0 = e^{4x}, \quad u^1 = 4e^{4x}, \quad u^2 = 16e^{4x}, \quad u^3 = 64e^{4x}$$

$$u^n = 4^n e^{4x} = u^5 = 4^5 e^{4x}$$

$$u = 1024 e^{4x}$$

$$y^{(n)} = u^n x^3 + n u^{n-1} 3x^2 + \frac{n(n-1)}{2} u^{n-2} \cdot 6x + \frac{n(n-1)(n-2)}{6} u^{n-3}$$

$$y^{(5)} = 1024e^{4x} x^3 + 5 \cdot 256 \cdot 3x^2 + 5(5-1)64e^{4x} + 5(5-1)(5-2)16e^{4x}$$

$$y^{(5)} = 1024e^{4x} x^3 + 3840x^2 + 1280e^{4x} + 960e^{4x}$$