

$$(1) \quad y = e^{x^2+x}$$

let

$$u = x^2 + x \quad ; y = e^u$$

$$\frac{du}{dx} = (2x+1) \quad ; \quad \frac{dy}{du} = e^u$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = (2x+1) \times e^u$$

$$\frac{dy}{dx} = e^u (2x+1) = e^{(x^2+x)} (2x+1)$$

$$y' = (2x+1) (e^{x^2+x})$$

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} (2x+1) (e^{x^2+x})$$

using product rule

$$\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$\text{let } u = (2x+1) \quad v = e^{x^2+x}$$

$$\frac{du}{dx} = 2$$

$$\frac{dv}{dx} = (2x+1) (e^{x^2+x})$$

$$\therefore \frac{d^2y}{dx^2} = (2x+1) \times (2x+1) e^{(x^2+x)} + e^{x^2+x} \times 2$$

$$\frac{d^2y}{dx^2} = (2x+1) e^{(x^2+x)} \times 2x+1 + 2e^{x^2+x}$$

$$\text{here } y' = (2x+1) e^{x^2+x} \quad \text{and } y = e^{x^2+x}$$

$$\frac{d^2y}{dx^2} = y' (2x+1) + 2y \quad \text{or } y'' = y' (2x+1) + 2y \quad \underline{\underline{Q.E.D}}$$

(b) $y'' = y'(2x+1) + 2y$ - (1) using Leibniz

$$w_1^{(n)} = u^n = y^{(n+2)}$$

$w_2^{(n)}$	u	v
$u^n = \frac{y^{(n)}}{y^{(1+n)}}$	$2x+1$	2
$u^{n-1} = \frac{y^{(n-1)}}{y^{(n)}}$	2	v^1
	0	v^2

$$w_2^{(n)} = u^n v' + n u^{n-1} v^2$$

$$w_2^{(n)} = (2x+1) y^{(1+n)} + n y^n$$

$$w_3^{(n)} = 2y^n$$

from eqn

$$y^{(n+2)} = (2x+1)y^{(1+n)} + n y^n + 2y^n$$

i.e $w_1^{(n)} = w_2^{(n)} + w_3^{(n)}$

$$y^{(n+2)} = y^{(1+n)}(2x+1) + (n+1) 2y^n =$$

(2) using Leibniz theorem

$$y = x^3 e^{4x} \quad \text{determine } y^{(n)}$$

let $u = e^{4x}$ $v = x^3$

$$u^n = 4^n e^{4nx} \quad v^1 = 3x^2$$

$$u^{n-1} = 4^{(n-1)} e^{4(n-1)x} \quad v^2 = 6x$$

$$u^{n-2} = 4^{(n-2)} e^{4(n-2)x} \quad v^3 = 6$$

$$u^{n-3} = 4^{(n-3)} e^{4(n-3)x} \quad v^4 = 0$$

Recall $y^{(n)} = u^n v + n u^{n-1} v^1 + \frac{n(n-1)}{2!} u^{n-2} v^2 + \frac{n(n-1)(n-2)}{3!} u^{n-3} v^3$

$$y^{(n)} = 4^n e^{4nx} x^3 + n 4^{(n-1)} e^{4(n-1)x} x^2 + \frac{n(n-1)}{2!} 4^{(n-2)} e^{4(n-2)x} x + \frac{n(n-1)(n-2)}{3!} 4^{(n-3)} e^{4(n-3)x}$$

$$y^{(5)} = 4^5 e^{4x} x^3 + 5 \cdot 4^4 e^{4x} x^2 + \frac{5 \cdot 4^3 e^{4x} x^2}{2!} + \frac{5x+5}{3!} 4^2 e^{4x} \cdot 6$$

$$y^{(5)} = 1024x^3 e^{4x} + 3840x^2 e^{4x} + 3840x e^{4x} + 96 e^{4x}$$

$$y^{(5)} = 64 e^{4x} (16x^3 + 60x^2 + 60x + 15)$$

(2x) $x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = 0$

$$x^2 y'' + x y' + y = 0$$

$$w_1^{(n)} = x^2 y''$$

let $u = y'' \rightarrow v = x^2$
 $u^n = y^{2+n} \rightarrow v' = 2x$
 $y^{n-1} = u^{(n+1)} \rightarrow v^2 = 2$
 $y^{n-2} = y^{(n)} \rightarrow v^3 = 0$

$$w_2^n = x y'$$

let $u = y' \rightarrow v = x$
 $u^n = y^{1+n} \rightarrow v' = 1$
 $u^{n-1} = y^n \rightarrow v^2 = 0$

$$w_3^n = y$$

let $u = y \rightarrow v = 1$
 $u^n = y^n \rightarrow v' = 0$

$$w_1^{(n)} = n! v + n(n-1) \frac{v^2}{2!} + n(n-1)(n-2) \frac{v^3}{3!}$$

$$w_1^{(n)} = y^{(n+2)} x^2 + n(n-1) y^{(n+1)} x + \frac{n(n-1)}{2!} y^{(n)} x^2$$

$$w_1^{(n)} = y^{(n+2)} x^2 + n y^{(n+1)} x + n(n-1) y^n$$

$$w_2^{\wedge} = u^{(n+1)} x^2 + y^n x$$

$$w_2^{(n)} = 2y^{(n+1)} + ny^n$$

$$w_3^{\wedge} = y^n$$

$$w_1^{(n)} + w_2^{(n)} + w_3^{\wedge} = 0$$

$$\therefore y^{(n+2)} x^2 + ny^{(n+1)} x + n(n-1)y^n + 2y^{(n+1)} + ny^n + y^n = 0$$

$$y^{(n+2)} x^2 + (n2x + x)y^{(n+1)} + (n(n-1) + n + 1)y^n = 0$$

$$x^2 y^{(n+2)} + (2n+1)x y^{(n+1)} + (n^2 - n + n + 1)y^n = 0$$

$$x^2 y^{(n+2)} + (2n+1)x y^{(n+1)} + (n^2 + 1)y^n = 0 \quad \text{Q.E.D.}$$