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ENG 381

Computer Engineering.

Question.

If $y = e^{x^2+x}$

show that

$y'' = y'(2x+1) + 2y$

and hence prove that $y^{(n+2)} = (2x+1)y^{(n+1)} + 2(n+1)y$

Soln.

$y = e^{x^2+x}$ ——— (1)

Using chain rule; let $u = x^2 + x$, $\frac{\delta u}{\delta x} = 2x+1$

$\Rightarrow y = e^u$, $dy/\delta u = e^u$

$\frac{\delta y}{\delta x} = \frac{\delta y}{\delta u} \times \frac{\delta u}{\delta x}$

$\frac{\delta y}{\delta x} = e^u(2x+1)$

$\delta y/\delta x = e^{x^2+x}(2x+1)$

$\delta y/\delta x = y' = e^{x^2+x}(2x+1$ ——— (2)

$d^2y/\delta x^2 = y''$

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Using product rule; $\frac{\delta y}{\delta x} = U \delta v/\delta x + V \delta u/\delta x$

$U = e^{x^2+x}$, $\delta u/\delta x = \delta u/\delta y \times \delta y/\delta x$

Let $y = x^2 + x$, $\delta y/\delta x = 2x+1$

$\Rightarrow u = e^y$, $\delta u/\delta y = e^y$

$\Rightarrow \delta u/\delta x = e^y \times (2x+1)$

$= (2x+1)\, e^{x^2+x}$

If $v = 2x+1$, $\frac{\delta v}{\delta x} = 2$

$\therefore y'' = \frac{\delta^2 y}{\delta x^2} = (e^{x^2+x})2 + (2x+1)(2x+1)\, e^{x^2+x}$

$y'' = 2(e^{x^2+x}) + (2x+1)(2x+1)e^{x^2+x}$

Recall;

from equ 1: $y = e^{x^2+x}$

Let $w_1 = y^{(2)}$

$w_2 = y^{(1)}(2x+1)$

$w_3 = 2y$

a $w_1 = y^{(2)}$

$\Rightarrow u = y^{(2)}$, $v = 1$

$\therefore u^n = y^{n+2}$, $v' = 0$

Hence,

$w_1^n = u^n v^0 + {}^nC_1 \cdot u^{n-1} v' + {}^nC_2 \cdot u^{n-2} v^2 + \ldots\ldots$

$w_1^n = u^n v^0 + n u^{n-1} v' + \frac{n(n-1)}{2} u^{n-2} v^2$

$w_1^n = y^{n+2} \cdot 1 + n y^{n-1+2} \cdot 0$

$\Rightarrow w_1^n = y^{n+2}$

b $w_2 = y^{(1)}(2x+1)$

$u = y'$, $v = 2x+1$

$u' = y''$, $v' = 2$, $v'' = 0$

$\Rightarrow u^n = y^{n+1}$

$w_2^n = u^n v^0 + n u^{n-1} v' + \frac{n(n-1)}{2} u^{n-2} v^2$

$w_2^n = y^{n+1}(2x+1) + n(y^{n+1-1})_n + 2 \cancel{(n-1)} + \frac{n(n-1)}{2} y^{n-2+1} \cdot 0$

$w_2^n = (2x+1) y^{n+1}, 2ny^n$

c $W_3 = 2y$

$u = y, \quad v = 2$

$y' = y', v' = 0$

$\Rightarrow u^n = y^n$

Hence

$$W_3^n = u^n v^0 + n u^{n-1} v' + \frac{n(n-1)}{2} u^{n-2} v^2$$

$$W_3^n = y^n \cdot 2 + n y^{n-1} \cdot 0$$

$$W_3^n = 2y^n$$

$$\therefore W_1^n = W_2^n + W_3^n$$

$$y^{(n+2)} = (2x+1) y^{(n+1)} + 2ny^n + 2y^n$$

$$y^{(n+2)} = (2x+1) y^{(n+1)} + 2(n+1) y^n$$

2 $y = x^3 e^{4x}$

$v = \cancel{x^3}\, x^3, \quad u = e^{4x}$

$v' = 3x^2, \quad u' = 4e^{4x}$

$v'' = 6x, \quad u'' = 16 e^{4x}$

$v''' = 6, \quad u''' = 64 e^{4x}$

$\therefore u^n = 4^n e^{4x}$

$$y^n = {}^nC_0 u^n v^0 + {}^nC_1 u^{n-1} v' + {}^nC_2 u^{n-2} v^2 + {}^nC_3 u^{n-3} v^3$$

Continuation.

$$y^n = u^n v^0 + n u^{n-1} v^1 + \frac{n(n-1)}{2!} v^2 u^{n-2} + \frac{n(n-1)(n-2)v^3 u^{n-3}}{3!}$$

$$y^n = u^n v^0 + n u^{n-1} v^1 + \frac{n(n-1) u^{n-2} v^2}{2!} + \frac{n(n-1)(n-2) v^3 u^{n-3}}{3!}$$

when $n=5$

$$y^{(5)} = (4^5 e^{4x} \cdots x^3) + (5 \cdot 4^{5-1} e^{4x} \cdot 3x^2) + \frac{5(5-1)4^{5-2} e^{4x} \cdot 6x}{2\times 1}$$

$$+ \frac{5(5-1)(5-2)4^{5-3} e^{4x} \cdot 6}{3\times 2\times 1}$$

$$y^{(5)} = 1024 e^{4x} x^3 + 3840 e^{4x} x^2 + 3840 e^{4x} x + 960 e^{4x}$$

$$y^{(5)} = 64 e^{4x} (16x^3 + 60x^2 + 60x + 15)$$

(iii) $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

Show that $x^2 y^{(n+2)} + (2n+1) x y^{(n+1)} + (n^2+1) y^{(n)} = 0$

$\Rightarrow x^2 y'' + xy' + y = 0$

Let $w_1 = x^2 y''$

So that $w_1^n + w_2^n + w_3^n = 0$

$v = x^2$, $u = y''$

$v' = 2x$, $u' = y'''$

$v'' = 2$, $u'' = y^{iv}$

$\therefore u^n = y^{n+2}$

$$w_1{}^n = u^n v^0 + n u^{n-1} v' + \frac{n(n-1)}{2} u^{n-2} v^2$$

$$= y^{n+2} \cdot x^2 + n(y^{n+2-1}) \cdot 2x + \frac{n(n-1)\, y^{n+2-2}}{2} \cdot 2$$

$$w_1{}^n = x^2 (y^{n+2}) + 2n\, x (y^{n+1}) + n(n-1) y^n$$

b Let $w_2 = xy'$

$v = x,\ u = y'$

$v' = 1,\ u' = y''$

$\therefore u^n = y^{n+1}$

$$w_2{}^n = u^n v^0 + n u^{n-1} v' + \frac{n(n-1)}{2} u^{n-2} v^2$$

$$= y^{(n+1)} x + n(y^{n+1-1}) \cdot 1$$

$$\Longrightarrow w_2{}^n = x y^{n+1} + n y^n$$

c Let $w_3 = y$

$u = y\ ,\ v = 1$

$u' = y'\ ,\ v' = 0$

$\therefore u^n = y^n$

$$w_3{}^n = u^n v^0 + n u^{n-1} v'$$

$$= y^n \cdot 1 + n y^{n-1} \cdot 0$$

$\therefore w_3{}^n = y^n$

From eqn *

$W_1^n + W_2^n + W_3^n = 0$

$x^2 y^{n+2} + 2nx(y^{n+1}) + n(n-1)y^n + xy^{n+1} + ny^n + y^n = 0$

Factorising, we have

$x^2 y^{n+2} + 2nx(y^{n+1}) + xy^{n+1} + n(n-1)y^n + ny^n + y^n = 0$

$x^2 y^{(n+2)} \quad xy^{n+1}(2n+1) + y^n(n^2 - n + n + 1) = 0$

$\therefore x^2 y^{(n+2)} + (2n+1)xy^{n+1} + (n^2+1)y^n = 0$