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1) If $y = e^{x^2+x}$
Show that
$y'' = y'(2x+1) + 2y$
or prove that $y^{(n+2)} = (2x+1)y^{n+1} + 2(n+1)y^{n}$

Solution

$y = e^{x^2+x}$ $\therefore$ $\ln y = x^2 + x$

Differentiating both sides

$\frac{1}{y}\frac{dy}{dx} = 2x+1$

multiplying by y

$dy/dx = (2x+1)\,y$

$\frac{d^2y}{dx^2} = V\,du/dx + u\,dv/dx$

$V = (x+1)$ $\qquad u = y$

$dv/dx = 2$ $\qquad du/dx = du/dy \cdot \frac{dy}{dx} = 1 \times dy/dx$

$\frac{du}{dy} = 1$

$\frac{du}{dx} = 1 \cdot \frac{dy}{dx} = \frac{dy}{dx}$

$\frac{d^2y}{dx^2} = (2x+1) \cdot \frac{dy}{dx} + 2y$

$d^2y/dx^2 = (2x+1)\,\frac{dy}{dx} + 2y$

$y'' = (2x+1)y' + 2y$

$y'' - (2x+1)y' + 2y = 0$

$y^{(2)} - (2x+1)y^{(1)} - 2y = 0$

for $y^{(2)}$

nth derivative $= y^{(n+2)}$

for $(2x+1)y'$

$u = y'$ $\qquad V = 2x+1$

$u^{n} = y^{(n+1)}$ $\qquad V^{(1)} = 2$

$u^{(n-1)} = y^{n}$ $\qquad V^{(2)} = 0$

Using Leibnitz Theorem

$y^{(n)} = u^{(n)}v + nu^{(n-1)}v^{(1)} + \dots$

for $2y$

n^{th} derivative $= 2y^{n}$

$w_1^{(n)} = y^{(n+2)}$

$w_2^{(n)} = y^{(n+1)}(2x+1) + y^{n}2n$

$w_3^{(n)} = 2y^{n}$

$y^{(n+2)} - y^{(n+1)}(2x+1) - y^{n}2n - 2y^{n} = 0$

$y^{(n+2)} = y^{(n+1)}(2x+1)\,2y^{n}(n+1)$ QED

2) Using Leibnitz theorem given that

(i) $y = x^3e^{4x}$, determine $y^{(5)}$

$y^{(n)} = u^{(n)}v + nu^{(n-1)}v^{(1)} + \frac{n(n-1)u^{(n-2)}v^{2}}{2!} + \frac{n(n-1)(n-2)u^{(n-3)}v^{3}}{3!}$

$u = e^{4x}$

$u^{n} = 4^{5}e^{4x}$

$u^{(n-1)} = 4^{(n-1)}e^{4x}$

$u^{(n-2)} = 4^{(n-2)}e^{4x}$

$u^{(n-3)} = 4^{(n-3)}e^{4x}$

$u^{n} = 4^{5}e^{4x} = 1024e^{4x}$

$u^{(n-1)} = 4^{(5-1)}e^{4x} = 256e^{4x}$

$u^{(n-2)} = 4^{(5-2)}e^{4x} = 64e^{4x}$

$u^{(n-3)} = 4^{(5-3)}e^{4x} = 16e^{4x}$

$v = x^3$

$v^{1} = 3x^2$

$v^{2} = 6x$

$v^{(3)} = 6$

$v = 0$

$y^{(5)} = 1024e^{4x}x^3 + 5\times256e^{4x}\times3x^2 + \frac{5(5-1)\times64e^{4x}\times6x}{2!} + \frac{5(5-1)(5-2)\times16e^{4x}}{3!}$

$y^{(5)} = 1024e^{4x}x^3 + 3840e^{4x}x^2 + \frac{7680e^{4x}x}{2!} + \frac{5760e^{4x}}{3!}$

$y^{(5)} = 1024e^{4x}x^3 + 3840e^{4x}x^2 + 3840e^{4x}x + 960e^{4x}$

$y^{(5)} = e^{4x}(1024x^3 + 3840x^2 + 3840x + 960)$

(ii) $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

It can be re written as

$x^2 y'' + xy' + y = 0$

$x^2 y^{(2)} + xy^{(1)} + y = 0$

for $x^2 y^{(2)}$

$u = y^{(2)}$ $\quad v = x^2$

$u^{(n)} = y^{(n+2)}$ $\quad v^{(1)} = 2x$

$u^{(n-1)} = y^{(n+1)}$ $\quad v^{(2)} = 2$

$u^{(n-2)} = y^{(n)}$ $\quad v^{(3)} = 0$

for xy'

$u = y'$ $\quad v = x$

$u^{(n)} = y^{(n+1)}$ $\quad v^{(1)} = 1$

$u^{(n-1)} = y^{(n)}$ $\quad v^{(2)} = 0$

$u^{(n-2)} = y^{(n-1)}$

for y

nth derivative $y^{(n)}$

Applying Leibnitz

$y^{(n)} = u^{(n)}v + nu^{(n-1)}v^{(1)} + \frac{n(n-1)u^{(n-2)}v^{2}}{2!}$

$w_1^{(n)} = y^{(n+2)}x^2 + ny^{(n+1)}2x + \frac{n(n-1)\cancel{(n-2)}y^{(n)}}{2!} + \frac{n(n-1)(n-2)y'}{3!}$

$w_2^{(n)} = y^{(n+1)}.x + ny^{(n)}.1 + \frac{n(n-1)y^{(n-1)}}{2!}.0$

$w_3^{(n)} = y^{(n)}$

$w_1^{(n)} + w_2^{(n)} + w_3^{(n)} = 0$

$x^2y^{(n+2)} + 2xny^{(n+1)} + n(n-1)y^{(n)} + xy^{(n+1)} + ny^{(n)} + y^{(n)} = 0$

Collecting like terms

$x^2y^{(n+2)} + xy^{(n+1)}(2n+1) + y^{(n)}(n^2 - n + n + 1) = 0$

$x^2y^{(n+2)} + xy^{(n+1)}(2n+1) + (n^2+1)y^{n} = 0$