1. $y = e^{x^2+x}$

let

$u = x^2 + x \quad ; y = e^u$

$\frac{du}{dx} = (2x+1) \quad ; \frac{dy}{du} = e^u$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = (2x+1) \times e^u$

$\frac{dy}{dx} = e^u(2x+1) = e^{(x^2+x)}(2x+1)$

$y' = \left(2x+1\right)\left(e^{x^2+x}\right)$

$\frac{d^2y}{dx^2} = \frac{dy}{dx}\left((2x+1)(e^{x^2+x})\right)$

using product rule

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

let $u = (2x+1) \quad v = e^{x^2+x}$

$\frac{du}{dx} = 2 \quad \frac{dv}{dx} = (2x+1)(e^{x^2+x})$

$\therefore \frac{d^2y}{dx^2} = (2x+1) \times (2x+1)e^{(x^2+x)} + e^{x^2+x} \times 2$

$\frac{d^2y}{dx^2} = (2x+1)e^{(x^2+x)} \times 2x+1 + 2e^{x^2+x}$

but $y' = (2x+1)e^{x^2+x}$ and $y = e^{x^2+x}$

$\frac{d^2y}{dx^2} = y'(2x+1) + 2y \Rightarrow y'' = y'(2x+1) + 2y$ Q.E.D

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$(x+y^{2})$

(b) $y'' = y'(2x+1) + 2y$ — ① using leibniz

$W_1^{(n)} = u^n = y^{(n+2)}$

$W_2^{(n)} =$ u $\quad$ v

y' $\quad$ $2x+1$

$u^n = y^{(1+n)}$ $\quad$ 2 $\quad$ v^1

$u^{n-1} = y^{(n)}$ $\quad$ 0 $\quad$ v^2

$W_2^{(n)} = u^n v' + n u^{n-1} v^1$

$W_2^{(n)} = (2x+1)\, y^{(1+n)} + n y^n 2$

$W_3^n = 2y^n$

from eqn

$y^{(n+2)} = (2x+1)(y^{1+n}) + n2y^n + 2y^n$

i.e. $W_1^{(n)} = W_2^{(n)} + W_3^{(n)}$

$y^{n+2} = y^{(1+n)}(2x+1) + (n+1)\, 2y^{(n)}$

(2) using leibniz theorem

$y = x^3 e^{4x}$ determine $y^{(5)}$

let $u = e^{4x}$ $\quad v = x^3$

$u^n = 4^n e^{4x}$ $\quad V_1' = 3x^2$

$u^{n-1} = 4^{(n-1)} e^{4x}$ $\quad V^{2} = 6x$

$u^{n-2} = 4^{(n-2)} e^{4x}$ $\quad V^3 = 6$

$u^{n-3} = 4^{(n-3)} e^{4x}$ $\quad V^{(4)} = 0$

recall $y^{(n)} = u^n v + n u^{n-1} v' + \frac{n(n-1)}{2!} u^{(n-2)} v^2 + \frac{n(n-1)(n-2)}{3!} u^{(n-3)} v^3$

$y^{(n)} = 4^n e^{4x} \times x^3 + n 4^{(n-1)} e^{4x} \times 3x^2 + \frac{n(n-1)}{2!} 4^{(n-2)} e^{4x} \times 6x$

$+ \frac{n(n-1)(n-2)}{3!} 4^{(n-3)} e^{4x} \times 6 + \cdots$

$$y^{(5)} = 4^5 e^{4x} x^3 + 5 \cdot 4^4 e^{4x} \cdot 3x^2 + \frac{5 \cdot 4}{2!} 4^3 e^{4x} \times 6x$$

$$+ \frac{5 \times 4 \times 3}{3!} 4^2 e^{4x} \cdot 6$$

$$y^{(5)} = 1024 x^3 e^{4x} + 3840 x^2 e^{4x} + 3840 x e^{4x} + 960 e^{4x}$$

$$y^{(5)} = 64 e^{4x} (16x^3 + 60x^2 + 60x + 15)$$

(2a) $$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

$$x^2 y'' + xy' + y = 0$$

$w_1^{(n)} = x^2 y''$

let $u = y''$ $\quad v = x^2$

$u^n = y^{2+n}$ $\quad v' = 2x$

$y^{n-1} = y^{(n+1)}$ $\quad v^2 = 2$

$y^{n-2} = y^{(n)}$ $\quad v^3 = 0$

$w_2^n = xy'$

let $u = y'$ $\quad v = x$

$u^n = y^{1+n}$ $\quad v' = 1$

$u^{n-1} = y^n$ $\quad v^2 = 0$

for $w_3^n = y$

let $u = y$ $\quad v = 1$

$u^n = y^n$ $\quad v' = 0$

$$w_1^{(n)} = u^n v + n u^{(n-1)} v' + \frac{n(n-1)}{2!} u^{n-2} v^2 + \frac{n(n-1)(n-2)}{3!} u^{(n-3)} v^3$$

$$w_1^{(n)} = y^{(n+2)} x^2 + n(y^{n+1}) 2x + \frac{n(n-1)}{\cancel{2!}} y^n \times \cancel{2}$$

$$w_1^{(n)} = y^{(n+2)} x^2 + n y^{n+1} 2x + n(n-1) y^n$$

$w_2^n = y^{(n+1)} x\, x + y^n \times 1$

$w_2^{(n)} = x y^{(n+1)} + n y^n$

$w_3^n = y^n$

$w_1^{(n)} + w_2^{(n)} + w_3^n = 0$

$\therefore\ y^{(n+2)} x^2 + n y^{n+1} 2x + n(n-1) y^n + x y^{(n+1)} + n y^n + y^n = 0$

$y^{(n+2)} x^2 + (n2x + x) y^{(n+1)} + (n(n-1) + n + 1) y^n = 0$

$x^2 y^{(n+2)} + (2n+1) x y^{(n+1)} + (n^2 - n + n + 1) y^n = 0$

$x^2 y^{(n+2)} + (2n+1) x y^{n+1} + (n^2 + 1) y^n = 0$ Q-E-D