① If $y = e^{x^2+x}$, show that

$y'' = y'(2x+1) + 2y$ and hence,

prove that $y^{(n+2)} = (2x+1)y^{(n+1)} + 2(n+1)y^{n}$

Soln

$y = e^{x^2+x}$

$y' = (2x+1)e^{x^2+x}$

$U = 2x+1 \quad U' = 2$

$V = e^{x^2+x} \quad V' = (2x+1)e^{x^2+x}$

$y'' = UV' + VU'$

$y'' = (2x+1)\left[(2x+1)e^{x^2+x}\right] + e^{x^2+x} \times 2$

but $y' = (2x+1)e^{x^2+x}$ & $y = e^{x^2+x}$

$y'' = y'(2x+1) - 2y$

$y'' = y'(2x+1) - 2u = 0$

Let $A = y''$

$U = y''$, $U^n = y^{n+2}$

$V = 1$, $V' = 0$

$y^n = U^n V + nV^{n-1}y'$

$y^n = y^{n+2} + ny^{n+1} \times 0$

$y^n = y^{n+2} \Rightarrow A$

Let $B = Y'(2x+1)$

$U = V'$ $\quad U^n = U^{n+1}$

$V = 2x+1$ $\quad V' = 2$ $\quad V'' = 0$

$Y^n = Y^{n+1}(2x+1) + nY^n \times 2$

$Y^n = Y^{n+1}(2x+1) + 2nY^n \Rightarrow B'$

$\cancel{U = Y}$ Let $C = 2U$

$U = Y$ $\quad U^n = Y^n$

$V = 2$ $\quad V' = 0$

$A - B - C = 0$

$A' - B' - C' = 0$

$Y^{n+2} = \left[Y^{n+1}(2x+1) + 2nY^n\right] - 2Y^n = 0$

$Y^{n+1} = Y^{n+1}(2x+1) + Y^n(2n+1)$

$Y^{n+1} = (2x+1)Y^{n+1} + 2(n+1)Y^n$

2 Using Leibnitz theorem

(a) $Y = \cancel{x^3 4x}\ x^3 e^{4x}$ determine Y^5

(b) $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = 0$ show that $x^2 y^{(n+2)} - (2n+1)xy^{(n+1)} + (n^2+1)y^n$.

Soln

(a) $U = e^{4x}$ $\quad U^n = 4^n e^{4x}$

$V = x^3$ $\quad V' = 3x^2$ $\quad V'' = 6x$ $\quad V''' = 5$

$y^n = 4^n e^{4x} \cdot x^3 + 3x^2 \cdot n \cdot 4^{n-1} e^{4x} + 3x(n^2-n)$
$+ n(n^2-3n+2) 4^{n-3} e^{4x}$

$y^5 = 1024 e^{4x} x^2 + 3 \cdot 256 \times 5 x^2 e^{4x}$

$y^5 = 1024 x^2 e^{4x} + 3840 x^2 e^{4x} + 3840 x + 960 e^{4x}$

$y^5 = 64 e^{4x} x^3 \left[16 + \frac{60}{x} + \frac{60}{x^2} + \frac{15}{x^3} \right]$

ii) $x^2 y'' + x y' + y = 0$

$y^n = u^n v + n u^{n-1} v' + \frac{n(n-1) u^{n-1} v^2}{2!}$
$+ \frac{n(n-1)(n-2)}{3!} \cdot u^{n-2} v^3$

~~$y = uv$~~

Let $A = x^2 y''$

$u = y''$ $\quad u^n = y^{n+2}$

$v = x^2$ $\quad v' = 2x$ $\quad v'' = 2$ $\quad v''' = 0$

~~$y^n = y^{n+2}$~~

$A' \Rightarrow y^n = y^{n+2} x^2 + n y^{n+1} \cdot 2x + n(n-1)$

Let $B = x y'$

$u = y'$ $\quad u^n = y^{n+1}$

$v = x$ $\quad v' = 1$, $v'' = 0$

$y^n = y^{n+1} \cdot x + n y^n$

Let $C = y$

$C' \Rightarrow y^n = y^n$

$A + B + C = 0$

$A' + B' + C' = 0$

$y^{n+2} x^2 + n y^{n+1} \cdot 2x + n(n-1) y^n + y^{n+1} \cdot x + n y^{n-1}$

$+ y^n = 0$

$y^{n+2} \cdot x^2 + y^{n+1} \cdot x(2n+1) + y^n(n^2 - \cancel{n} + \cancel{n} + 1) = 0$

$y^{n+2} \cdot x^2 + y^{n+1} \cdot x(2n+1) + y^n(n^2+1) = 0$