1.) $y'' = y'(2x+1) + 2y$

for $y'' \Rightarrow y^{(n+2)} + ny^{(n+1)} \cdot 0$

$= y^{(n+2)}$

for $\underset{u}{y'}\ \underset{v}{(2x+1)}$

$y'(2x+1) \Rightarrow y^{(n+1)}(2x+1) + ny^{n}(2)$

for $2y \Rightarrow 2y^{n}$

$\therefore$ we have;

$y^{(n+2)} = y^{(n+1)}(2x+1) + 2ny^{n} + 2y^{n}$

$\Rightarrow y^{(n+2)} = (2x+1)y^{(n+1)} + 2(n+1)y^{n}$.

2.) i, $y = \underset{v}{x^3}\ \underset{u}{e^{4x}}$

$u = e^{4x}$, $u' = 4e^{4x}$, $u'' = 16e^{4x}$, $u''' = 64e^{4x}$, $u^{(4)} = 256e^{4x}$ $u^{(5)} = 1024e^{4x}$

$v = x^3$, $v' = 3x^2$, $v'' = 6x$, $v''' = 6$, $v^{(4)} = 0$.

$y^{(5)} = u^{(5)}v + 5u^{(4)}v^{(1)} + 10u^{(3)}v^{(2)} + 10u^{(2)}v^{(3)} + 5u^{(1)}v^{(4)} + uv^{(5)}$

$y^{(5)} = 1024e^{4x} \cdot x^3 + [5(256e^{4x} \cdot 3x^2)] + [10(64e^{4x} \cdot 6x)] + [10(16e^{4x} \cdot 6)]$

$+ [5(4e^{4x} \cdot 0)] + 0$

$$y^{(5)} = 1024e^{4x}\cdot x^3 + 3840e^{4x}\cdot x^2 + 3840e^{4x}\cdot x + 960e^{4x}$$

$$\Rightarrow e^{4x}(1024x^3 + 3840x^2 + 3840x + 960)$$

$$y^{(n)} = 64e^{4x}(16x^3 + 60x^2 + 60x + 15)$$

ii) $x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = 0$

$\Rightarrow x^2y'' + xy' + y = 0$

Using leibnitz theorem.

from x^2y'', $u = y''$ & $x^2 = v$

$$x^2y'' = {}^nC_0 u^{(n)} v + {}^nC_1 u^{(n-1)} v^{(1)} + {}^nC_2 u^{(n-2)} v^{(2)}$$

$$\Rightarrow y^{(n+2)} x^2 + n y^{(n+1)}\cdot 2x + \frac{n(n-1)}{\cancel{2}!} y^{(n)}\cdot 2$$

$$x^2y'' \Rightarrow y^{(n+2)} x^2 + 2xn y^{(n+1)} + n(n-1)y^{(n)}$$

For xy': $u = y'$ and $v = x$

$$xy' = y^{(n+1)}x + ny^{(n)}$$

For $y \Rightarrow y^{(n)}$

$\therefore$ we have;

$$y^{(n+2)}x^2 + ny^{(n+1)}2x + n(n-1)y^{(n)} + y^{(n+1)}x + ny^{(n)} + y^{(n)}$$

$$y^{(n+2)}x^2 + xy^{(n+1)}(2n+1) + y^{(n)}[n(n-1)(n+1)]$$

$$y^{(n+2)}x^2 + xy^{(n+1)}(2n+1) + y^{(n)}[n(n^2-1)]$$

$$\Rightarrow x^2y^{(n+2)} + xy^{(n+1)}(2n+1) + y^{(n)}(n^3-n)$$