Mufutau Yusuf Adewola
16/ENG06/034
Mechanical Engineering
ENG 381 Engineering Maths II

If $y = e^{x^2+x}$
Show that
$y'' = y'(2x+1) + 2y$ and hence prove that
$y^{(n+2)} = (2x+1)y^{(n+1)} + 2(n+1)y^{n}$

Answer
$y = e^{x^2+x}$
$\ln y = x^2 + x$
Differentiating both sides
$\frac{1}{y}\, dy/dx = 2x+1$
multiply both sides by y.
$dy/dx = (2x+1)y$.

$\frac{d^2y}{dx^2} = u\frac{dv}{dx} + v\frac{du}{dx}$.
$v = 2x+1 \qquad dv/dx = 2$
$u = y \qquad du/dx = \frac{du}{dy} \cdot \frac{dy}{dx} = 1 \cdot \frac{dy}{dx}$.

So that;
$\frac{d^2y}{dx^2} = (2x+1) \cdot 1\frac{dy}{dx} + 2y$
$\frac{d^2y}{dx^2} = (2x+1)\frac{dy}{dx} + 2y$

$y'' = y'(2x+1) + \underline{2y}$

$y'' = y'(2x+1) + 2y.$

$y^{(2)} = y^{(1)}(2x+1) + 2y.$

Differentiating $y^{(1)}(2x+1)$

let $v = 2x+1$ $\quad u = y'$

$v' = 2$ $\quad u^{n} = y^{(n+1)}$

$v^{2} = 0$ $\quad u^{(n-1)} = y^{n}.$

Recall from leibnitz theorem.

$u^{n}v + nu^{(n-1)}v' + \cancel{n(n-1)u^{(n-2)}v^{2}}$ but $v^{2} = 0$

$= y^{(n+1)}\cdot(2x+1) + n(y^{n})\cdot 2.$

Differentiating $y^{(2)}$ we have; $y^{(n+2)}$

$(2y = 2y^{n})$

$y^{(n+2)} = y^{(n+1)}\cdot(2x+1) + 2ny^{n} + 2y^{n}.$

$y^{(n+2)} = (2x+1)y^{(n+1)} + 2y^{n}(n+1)$

Proved.

(2) Using Leibnitz theorem, given that

(i) $y = x^{3}e^{4x}$, determine $y^{(5)}$

$u = e^{4x}$ $\quad v = x^{3}$

$u^{n} = 4^{n}e^{4x}$ $\quad v' = 3x^{2}$

$u^{(n-1)} = 4^{(n-1)}e^{4x}$ $\quad v^{2} = 6x$

$u^{(n-2)} = 4^{(n-2)}e^{4x}$ $\quad v^{3} = 6$

$u^{(n-3)} = 4^{(n-3)}e^{4x}$ $\quad v^{4} = 0.$

where $n = 5$

$4^{5}e^{4x} = 1024e^{4x}$

$u^{(n-1)} = 4^{(5-1)}e^{4x} = 256e^{4x}$

$u^{(n-2)} = 4^{(5-2)}e^{4x} = 64e^{4x}$

$u^{(n-3)} = 4^{(5-3)}e^{4x} = 16e^{4x}$

$$y^{(n)} = (1024e^{4x}\cdot x^3) + n(256e^{4x}\cdot 3x^2) + \frac{n(n-1)}{2!}(64e^{4x}\cdot 6x$$
$$+ \frac{n(n-1)(n-2)}{3!}(16e^{4x})\cdot 6$$

We have;

$= (1024x^3 + 13840x^2 + 3840x + 960)e^{4x}$.

$$y^{(n)} = e^{4x}(1024x^3 + 3840x^2 + 3840x + 960)$$

2(ii) $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

Show that $x^2y^{(n+2)} + (2n+1)xy^{(n+1)} + (n^2+1)y^n = 0$

$x^2y'' + xy' + y = 0$

$u = y^{(n)}$ $\quad V = x^2$

$u^n = y^{(n+2)}$ $\quad V^1 = 2x$ $\Rightarrow y^{(n+2)}(x^2) + n(y^{n+1})2x + \frac{n(n-1)y^n(2)}{2!}$

$u^{(n-1)} = y^{(n+1)}$ $\quad V^2 = 2$

$u^{(n-2)} = y^n$ $\quad V^3 = 0$

Since $V^3 = 0$

for (xy')

$V = x$ $\quad u = y'$

$V' = 1$ $\quad u' = y^{(n+1)}$

$V^2 = 0$ $\quad u^{(n-1)} = y^n$.

Applying.

$$y^n = U^nV + nU^{(n-1)}V^{(1)} + \frac{n(n-1)}{2!}U^{(n-2)}V^2 + \dots \rightarrow 0.$$

(Since $V^2 = 0$)

~~$y^n = y^{(n+2)}(x^2) + n(y^{n+1})(2x) + n(n-1)y$~~

for (xy')

$y^n = y^{(n+1)}\cdot x + ny^{(n)}$

$$y^n = x^2 y^{(n+2)} + 2x\,n y^{(n+1)} + n(n-1) y^n + x y^{(n+1)} + n y^n - y^n$$

then

$$y^n = x^2 y^{(n+2)} + x y^{(n+1)} \cdot (2n+1) + y^n (n^2 - n + n + 1) = 0$$

$$= x^2 y^{(n+2)} + x y^{(n+1)} (2n+1) + y^n (n^2+1) = 0$$

$$y^n = x^2 y^{(n+2)} + x y^{(n+1)} (2n+1) + y^n (n^2+1) = 0$$